Differential Equations

Peter Juhasz

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Exercise 5.13

Exercise

Find all maximal solutions of the equations

$$\dot{y} = \frac{y}{t} - 2y^2 \qquad \text{Hint: Exercise 4.18} \qquad \dot{y} = -2y^2 t$$
$$\dot{y} = \frac{2y}{t} + \left(\frac{y}{t}\right)^2 \qquad \text{Hint: Example 1.5} \qquad \dot{y} = y^2$$

defined on $\mathcal{D}_X = \mathbb{R}_+ \times \mathbb{R}$.

Solution (first equation)

Let us treat first the first equation. The goal is, according to the hint, to transform the equation to the form of $\dot{y}_\psi = -2y_\psi^2 t$. Thus, we would like to use a linear spatial transformation so that

$$X_\psi(t, y_\psi) = -2y_\psi^2 t. \tag{1}$$

According to Definition 5.12, the relationship between the original and the transformed solutions is

$$y(t) = \Phi(t) y_\psi(t) + g(t), \tag{2}$$

where $\Phi(t)$ in our 1-dimensional case is a scalar valued function. To get $X_\psi(t, y_\psi)$, we can use formula (5.11) from the book (which is derived by substituting the above expression into the original ODE):

$$X_\psi(t, y_\psi) = \Phi^{-1}\left(-\dot{\Phi} y_\psi - \dot{g} + X\left(t, \Phi y_\psi + g\right)\right) = \Phi^{-1}\left(-\dot{\Phi} y_\psi - \dot{g} + \frac{\Phi y_\psi + g}{t} - 2(\Phi y_\psi + g)^2\right)$$

where we neglected the t arguments for simplicity, and used that $X(t, y) = y/t - 2y^2$. We bear in mind that our goal is to get back (1). Looking at the squared term we can guess that

$$g(t) = 0 \qquad \Phi(t) = t \qquad \mathcal{I}_\psi = \mathbb{R}_+,$$

so we can define the linear spatial transformation as

$$(\mathbb{R}_+,\ t\, y(t)),$$

which fulfills all criteria of Definition 5.12 of linear spatial transformation, as $\Phi(t) = t$ is invertible on $\mathbb{R}_+$. The transformed equation is the same as the ODE in Exercise 4.18:

$$\dot{y}_\psi = X_\psi(t, y_\psi) \qquad X_\psi(t, y_\psi) = -2y_\psi^2 t$$

Now we use formula 5.12 to define the region.

$$\mathcal{D}_{X_\psi} = \{(t,\ y_\psi) \in \mathcal{I}_\psi \times \mathbb{F}^n \,|\, (t, \Phi y_\psi + g) \in \mathcal{D}_X\} = \{(t,\ y_\psi) \in \mathbb{R}_+ \times \mathbb{R} \,|\, (t, t y_\psi) \in \mathbb{R}_+ \times \mathbb{R}\} = \mathcal{D}_X$$

The transformed equation is the same as the ODE in Exercise 4.18, and can be solved the same way. The only difference is the region of the solution: now we only need to consider positive t values as $\mathcal{D}_{X_\psi} = \mathbb{R}_+ \times \mathbb{R}$. The maximal solutions of the transformed ODE are in the form of $(I,\ y_\psi)$, where I and y_ψ are the following:

					$I = \mathbb{R}_+$	$y_\psi(t) = 0$
if	$t_0^2 \le \frac{1}{\eta}$	and	$\eta \neq 0$	then	$I = \mathbb{R}_+$	$y_\psi(t) = \frac{1}{t^2 - t_0^2 + 1/\eta}$
if	$t_0^2 > \frac{1}{\eta}$	and	$\eta < 0$	then	$I = \left(0, \sqrt{t_0^2 - \frac{1}{\eta}}\right)$	$y_\psi(t) = \frac{1}{t^2 - t_0^2 + 1/\eta}$
if	$t_0^2 > \frac{1}{\eta}$	and	$\eta > 0$	then	$I = \left(\sqrt{t_0^2 - \frac{1}{\eta}}, \infty\right)$	$y_\psi(t) = \frac{1}{t^2 - t_0^2 + 1/\eta}$

where in all cases $\eta \in \mathbb{R}$ and $t_0 \in \mathbb{R}_+$.

Now we need to transform the solution and the domains back to the original case. Let us begin with the solution:

$$y(t) = \Phi(t)y_\psi(t) + g(t) = t\, y_\psi(t).$$

To get back the domain we use formula 5.14 from the book:

$$\mathcal{D}_{(X_\psi)_{\psi^{-1}}} = (\mathcal{I}_\psi \times \mathbb{F}^n) \cap \mathcal{D}_X = (\mathbb{R} \times \mathbb{R}) \cap (\mathbb{R}_+ \times \mathbb{R}) = \mathcal{D}_X.$$

So for the sake of completeness, the maximal solutions are the following:

$$\begin{array}{llllllll}
 & & & & & I = \mathbb{R}_+ & & y(t) = 0 \\
\text{if} & t_0^2 \le \dfrac{1}{\eta} & \text{and} & \eta \neq 0 & \text{then} & I = \mathbb{R}_+ & & y(t) = \dfrac{t}{t^2 - t_0^2 + 1/\eta} \\
\text{if} & t_0^2 > \dfrac{1}{\eta} & \text{and} & \eta < 0 & \text{then} & I = \left(0, \sqrt{t_0^2 - \dfrac{1}{\eta}}\right) & & y(t) = \dfrac{t}{t^2 - t_0^2 + 1/\eta} \\
\text{if} & t_0^2 > \dfrac{1}{\eta} & \text{and} & \eta > 0 & \text{then} & I = \left(\sqrt{t_0^2 - \dfrac{1}{\eta}}, \infty\right) & & y(t) = \dfrac{t}{t^2 - t_0^2 + 1/\eta}
\end{array}$$

Solution (second equation)

Now we can move on to the next equation, where the goal is to transform the equation to the form of $\dot{y}_\psi = y_\psi^2$. We use a linear spatial transformation so that

$$X_\psi(t, y_\psi) = y_\psi^2. \tag{3}$$

Substituting the formula (5) from Definition 5.12 into the original ODE, or using formula 5.11 again:

$$X_\psi(t, y_\psi) = \Phi^{-1}\left(-\dot{\Phi}y_\psi - \dot{g} + X\left(t, \Phi y_\psi + g\right)\right) = \Phi^{-1}\left(-\dot{\Phi}y_\psi - \dot{g} + 2\,\frac{\Phi y_\psi + g}{t} + \frac{(\Phi y_\psi + g)^2}{t^2}\right)$$

where we used that $X(t, y) = 2y/t + (y/t)^2$. To get y_ψ^2 on the right-hand side, we choose

$$g(t) = 0 \qquad \Phi(t) = t^2 \qquad \mathcal{I}_\psi = \mathbb{R}_+,$$

so we can define the linear spatial transformation as

$$\left(\mathbb{R}_+,\ t^2\, y(t)\right),$$

which fulfills all criteria of Definition 5.12 of linear spatial transformation, as $\Phi(t) = t^2$ is invertible on $\mathbb{R}_+$. The transformed equation is:

$$\dot{y}_\psi = X_\psi(t, y_\psi) \qquad X_\psi(t, y_\psi) = y_\psi^2$$

Using formula 5.12 to define the region:

$$\mathcal{D}_{X_\psi} = \{(t,\, y_\psi) \in \mathcal{I}_\psi \times \mathbb{F}^n \,|\, (t, \Phi y_\psi + g) \in \mathcal{D}_X\} = \left\{(t,\, y_\psi) \in \mathbb{R}_+ \times \mathbb{R} \,|\, (t, t^2 y_\psi) \in \mathbb{R}_+ \times \mathbb{R}\right\} = \mathcal{D}_X$$

The transformed equation can be solved as in Example 1.5 or as a separable equation, the only difference again is the region of the solution as before. The maximal solutions of the transformed ODE are in the form of $(I,\, y_\psi)$, where I and y_ψ are the following:

$$\begin{array}{lllll}
 & & & I = \mathbb{R}_+ & y_\psi(t) = 0 \\
\text{if} & \eta > 0 & \text{then} & I = (\eta, \infty) & y_\psi(t) = \dfrac{1}{\eta - t} \\
\text{if} & \eta > 0 & \text{then} & I = (0, \eta) & y_\psi(t) = \dfrac{1}{\eta - t} \\
\text{if} & \eta \le 0 & \text{then} & I = \mathbb{R}_+ & y_\psi(t) = \dfrac{1}{\eta - t}
\end{array}$$

where in all cases $\eta \in \mathbb{R}$.

To transform the solution and the domains back to the original case, we have

$$y(t) = \Phi(t) y_\psi(t) + g(t) = t^2\, y_\psi(t).$$

To get back the domain we use formula 5.14 from the book:

$$\mathcal{D}_{(X_\psi)_{\psi^{-1}}} = (\mathcal{I}_\psi \times \mathbb{F}^n) \cap \mathcal{D}_X = (\mathbb{R}_+ \times \mathbb{R}) \cap (\mathbb{R}_+ \times \mathbb{R}) = \mathcal{D}_X.$$

For the sake of completeness, the maximal solutions are the following:

$$\begin{array}{llllll} & & & I = \mathbb{R}_+ & & y(t) = 0 \\ \text{if} & \eta > 0 & \text{then} & I = (\eta, \infty) & & y(t) = \dfrac{t^2}{\eta - t} \\ \text{if} & \eta > 0 & \text{then} & I = (0, \eta) & & y(t) = \dfrac{t^2}{\eta - t} \\ \text{if} & \eta \leq 0 & \text{then} & I = \mathbb{R}_+ & & y(t) = \dfrac{t^2}{\eta - t} \end{array}$$

Exercise 6.7

Exercise

1. Let C, $D \in \mathrm{M}_n(\mathbb{C})$ with C invertible. Show that $e^{CDC^{-1}} = Ce^D C^{-1}$.
2. Let $A \in \mathrm{M}_n(\mathbb{C})$ be diagonalizable. Denote by $\eta_1, \ldots, \eta_n$ a basis of eigenvectors with the corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. Put $C = (\eta_1, \ldots, \eta_n)$. Argue that

$$e^A = C \,\mathrm{Diag}\left[e^{\lambda_1}, \ldots, e^{\lambda_n}\right] C^{-1}.$$

3. For $c \in \mathbb{R}$ define a 2×2 matrix by

$$A_c = \begin{bmatrix} 2 & 1 \\ c & 2 \end{bmatrix}.$$

   Compute e^{A_c} for the two cases $c = -4$ and $c = 4$.

Solution (part 1)

We can use the definition of the exponential function $e^A = \sum_{k=0}^{\infty}(A^k/k!)$:

$$e^{CDC^{-1}} = \sum_{k=0}^{\infty} \frac{(CDC^{-1})^k}{k!} = \sum_{k=0}^{\infty} \frac{CDC^{-1} \cdots CDC^{-1}}{k!} = \sum_{k=0}^{\infty} \frac{CD^kC^{-1}}{k!} = C\left(\sum_{k=0}^{\infty} \frac{D^k}{k!}\right) C^{-1} = Ce^D C^{-1}$$

Solution (part 2)

If A is diagonalizable, then it can be written in the following form: $A = CDC^{-1}$, where $D = \mathrm{Diag}\left[\lambda_1, \ldots, \lambda_n\right]$ and C contains the eigenvectors in its columns. Using the previous part of the exercise, we have:

$$e^A = e^{CDC^{-1}} = Ce^D C^{-1} = C \,\mathrm{Diag}\left[e^{\lambda_1}, \ldots, e^{\lambda_n}\right] C^{-1},$$

where in the last step we have used Example 6.4 from the book.

Solution (part 3)

We would like to diagonalize A, so first, we determine the eigenvalues.

$$|A - \lambda I| = \left|\begin{bmatrix} 2-\lambda & 1 \\ c & 2-\lambda \end{bmatrix}\right| = 0 \qquad \Longrightarrow \qquad \lambda_{1;2} = 2 \pm \sqrt{c}$$

If $c = 4$, then $\lambda_1 = 0$ and $\lambda_2 = 4$. We calculate the eigenvectors:

$$\lambda_1 = 0 \qquad \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} \eta_{11} \\ \eta_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} \eta_{11} \\ \eta_{21} \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$\lambda_2 = 4 \qquad \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} \eta_{12} \\ \eta_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} \eta_{12} \\ \eta_{22} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Using the eigenvectors we can construct C and also calculate its inverse.

$$C = \begin{bmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} \qquad C^{-1} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}$$

Now we use the results from part 2 to calculate the exponential:

$$e^{A_4} = C \,\mathrm{Diag}\left[e^0, \ldots, e^4\right] C^{-1} = \frac{1}{4}\begin{bmatrix} 2(1+e^4) & 1-e^4 \\ 4(1-e^4) & 2(1+e^4) \end{bmatrix}.$$

If $c = -4$, then $\lambda_1 = 2(1+i)$ and $\lambda_2 = 2(1-i)$. We calculate the eigenvectors:

$$\lambda_1 = 2(1+i) \qquad \begin{bmatrix} -2i & 1 \\ 4 & -2i \end{bmatrix} \begin{bmatrix} \eta_{11} \\ \eta_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} \eta_{11} \\ \eta_{21} \end{bmatrix} = \begin{bmatrix} -i \\ 2 \end{bmatrix}$$

$$\lambda_2 = 2(1-i) \qquad \begin{bmatrix} 2i & 1 \\ 4 & 2i \end{bmatrix} \begin{bmatrix} \eta_{12} \\ \eta_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \Longrightarrow \qquad \begin{bmatrix} \eta_{12} \\ \eta_{22} \end{bmatrix} = \begin{bmatrix} i \\ 2 \end{bmatrix}$$

Using the eigenvectors we can construct C and also calculate its inverse.

$$C = \begin{bmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{bmatrix} = \begin{bmatrix} -i & i \\ 2 & 2 \end{bmatrix} \qquad C^{-1} = \frac{1}{4} \begin{bmatrix} 2i & 1 \\ -2i & 1 \end{bmatrix}$$

Now we use the results from part 2 to calculate the exponential:

$$e^{A_{-4}} = C \, \mathrm{Diag}\left[e^{2(1+i)}, \ldots, e^{2(1-i)}\right] C^{-1} = \frac{e^2}{4} \begin{bmatrix} 2(e^{2i} + e^{-2i}) & -i(e^{2i} - e^{-2i}) \\ 4i(e^{2i} - e^{-2i}) & 2(e^{2i} + e^{-2i}) \end{bmatrix}.$$

We can recognize that the exponential forms of sine and cosine appeared:

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \qquad \sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}.$$

So, the simplified solution:

$$e^{A_{-4}} = \frac{e^2}{2} \begin{bmatrix} 2\cos 2 & -\sin 2 \\ -4\sin 2 & 2\cos 2 \end{bmatrix}.$$

Exercise 6.9

Exercise

Let
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Compute e^{tA} and e^{tB} using Example 6.9 and find the maximal solutions of the ODE-s $\dot{y} = Ay$ and $\dot{y} = By$, satisfying $y(0) = (1, 0)$.

Solution

As A, $B \in \mathrm{M}_2(\mathbb{R})$ and $\mathrm{Tr}(A) = \mathrm{Tr}(B) = 0$, all conditions are fulfilled to use Example 6.9. We calculate the determinants:
$$|A| = 1 \qquad |B| = -1.$$

Use formulas (6.8) and (6.9) to get e^{tA} and e^{tB}, respectively.

$$|A| > 0 \qquad e^{tA} = \cos\left(t\sqrt{|A|}\right) I + \frac{\sin\left(t\sqrt{|A|}\right)}{\sqrt{|A|}} A = \cos t\, I + \sin t\, A = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$
$$|B| < 0 \qquad e^{tB} = \operatorname{ch}\left(t\sqrt{-|B|}\right) I + \frac{\operatorname{sh}\left(t\sqrt{-|B|}\right)}{\sqrt{|B|}} B = \operatorname{ch} t\, I + \operatorname{sh} t\, B = \begin{bmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{bmatrix}$$

According to Theorem 6.12, the maximal solutions of the two ODE-s $\dot{y} = Ay$ and $\dot{y} = By$ are the following:

$$\left\{(\mathbb{R}, e^{tA}\eta_A) \,|\eta_A \in \mathbb{R}^2\right\} = \left\{\left(\mathbb{R}, \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \eta_A\right) |\eta_A \in \mathbb{R}^2\right\}$$
$$\left\{(\mathbb{R}, e^{tB}\eta_B) \,|\eta_B \in \mathbb{R}^2\right\} = \left\{\left(\mathbb{R}, \begin{bmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{bmatrix} \eta_B\right) |\eta_B \in \mathbb{R}^2\right\}$$

To fulfill the IVP of $y(0) = (1, 0)$,

$$\begin{aligned} \eta_{A1}\cos t + \eta_{A2}\sin t = 1 \qquad &\Longrightarrow \qquad \eta_{A1} = 1 \\ -\eta_{A1}\sin t + \eta_{A2}\cos t = 0 \qquad &\Longrightarrow \qquad \eta_{A2} = 0 \end{aligned}$$

and for η_B:

$$\begin{aligned} \eta_{B1}\operatorname{ch} t + \eta_{B2}\operatorname{sh} t = 1 \qquad &\Longrightarrow \qquad \eta_{B1} = 1 \\ \eta_{B1}\operatorname{sh} t + \eta_{B2}\operatorname{ch} t = 0 \qquad &\Longrightarrow \qquad \eta_{B2} = 0. \end{aligned}$$

Thus, the maximal solutions satisfying the IVP are

$$\dot{y} = Ay \qquad \left(\mathbb{R}, \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}\right)$$
$$\dot{y} = By \qquad \left(\mathbb{R}, \begin{bmatrix} \operatorname{ch} t \\ \operatorname{sh} t \end{bmatrix}\right)$$

Exercise 6.11

Exercise

Compute e^{tA}, e^{tB}, e^{tC}, e^{tD} where

$$A = \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix} \qquad D = \begin{bmatrix} 5 & 4 \\ -2 & -1 \end{bmatrix}$$

Solution

We can use Remark 6.10 to decompose the matrices to a sum where one of the components has trace 0, and the other is diagonal:

$$M = \underbrace{M - \frac{1}{2}\text{Tr}(M)\, I}_{M_1,\ \text{Tr}(M_1)=0} + \underbrace{\frac{1}{2}\text{Tr}(M)\, I}_{M_2,\ \text{diagonal matrix}}$$

Then, by Proposition 6.1,

$$e^M = e^{M_1+M_2} = e^{M_1} \cdot e^{M_2},$$

and e^{M_1} can be calculated from Example 6.9, whereas e^{M_2} can be calculated by Example 6.4.

$$\text{Tr}(A) = 0, \qquad A_1 = A = \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad |A_1| = 4 > 0$$

Now, by using formula (6.8),

$$e^{tA} = \cos 2t\, I + \frac{1}{2}\sin 2t\, A_1 = \begin{bmatrix} \cos 2t + \sin 2t & 2\sin 2t \\ -\sin 2t & \cos 2t - \sin 2t \end{bmatrix}$$

$$\text{Tr}(B) = 0, \qquad B_1 = B = \begin{bmatrix} 3 & 4 \\ -2 & -3 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad |B_1| = -1 < 0$$

Now, by using formula (6.9),

$$e^{tB} = \text{ch}\, t\, I + \text{sh}\, t\, B_1 = \begin{bmatrix} \text{ch}\, t + 3\,\text{sh}\, t & 4\,\text{sh}\, t \\ -2\,\text{sh}\, t & \text{ch}\, t - 3\,\text{sh}\, t \end{bmatrix}$$

$$\text{Tr}(C) = 2, \qquad C_1 = \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix} = A \qquad C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, by using e^{tA},

$$e^{tC} = e^{tA} \cdot e^t I = e^t \begin{bmatrix} \cos 2t + \sin 2t & 2\sin 2t \\ -\sin 2t & \cos 2t - \sin 2t \end{bmatrix}$$

$$\text{Tr}(D) = 4, \qquad D_1 = \begin{bmatrix} 3 & 4 \\ -1 & -3 \end{bmatrix} = B \qquad D_2 = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Now, by using e^{tB},

$$e^{tD} = e^{tB} \cdot e^{2t} I = e^{2t} \begin{bmatrix} \text{ch}\, t + 3\,\text{sh}\, t & 4\,\text{sh}\, t \\ -2\,\text{sh}\, t & \text{ch}\, t - 3\,\text{sh}\, t \end{bmatrix}$$

Exercise 6.17

Exercise

Compute e^{tA} and e^{tB} where

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 4 & 4 & 0 \\ 1 & 2 & 2 \\ -1 & -2 & 2 \end{bmatrix}$$

Solution

We can follow Example 6.18. First, we compute the eigenvalues:

$$\begin{aligned} |A - \mu I| &= (2-\mu)(1-\mu)^2 & \Longrightarrow \quad & \mu_1 = 1 \quad \mu_2 = 2 \\ |B - \mu I| &= (4-\mu)(2-\mu)^2 \underbrace{-8 + 4(4-\mu) - 4(2-\mu)}_{0} & \Longrightarrow \quad & \mu_1 = 2 \quad \mu_2 = 4 \end{aligned}$$

Here μ_1 has multiplicity 2 in both cases. We follow Case II. from Example 6.18.

$$\begin{aligned} e^{tA} &= -e^t \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} - (1+t)e^t \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix} + e^{2t} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}^2 = \\ &= e^t \begin{bmatrix} e^t & e^t - 1 & e^t - 1 - t \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} e^{tB} &= -2e^{2t} \begin{bmatrix} 0 & 4 & 0 \\ 1 & -2 & 2 \\ -1 & -2 & -2 \end{bmatrix} - \frac{1+2t}{4} e^{2t} \begin{bmatrix} 2 & 4 & 0 \\ 1 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 4 & 0 \\ 1 & -2 & 2 \\ -1 & -2 & -2 \end{bmatrix} + \frac{1}{4} e^{4t} \begin{bmatrix} 2 & 4 & 0 \\ 1 & 0 & 2 \\ -1 & -2 & 0 \end{bmatrix}^2 = \\ &= e^{2t} \left(e^{2t} \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} + t \begin{bmatrix} -2 & 0 & -4 \\ 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 & -16 & -4 \\ -3 & 8 & -6 \\ 5 & 8 & 10 \end{bmatrix} \right) \end{aligned}$$

Exercise 6.18

Exercise

Let $c \in \mathbb{C}$. Consider the matrix $A_c \in \mathrm{M}_2(\mathbb{C})$ given by:

$$A_c = \begin{bmatrix} 1 & c \\ -1 & -1 \end{bmatrix}.$$

Compute e^{tA_c} for $c = 0,\ 1,\ 2,\ 1+2i,\ 1-2i$.

Solution: $c = 2$

If $c = 2$, then all elements of the matrix are real. As $\operatorname{Tr} A_2 = 0$, we can use Example 6.9 to calculate the solution. The determinant is $|A_2| = -1 + c = 1 > 0$, so using formula (6.8) we have

$$e^{tA_2} = \cos t\, I + \sin t\, A_2 = \begin{bmatrix} \cos t + \sin t & 2\sin t \\ -\sin t & \cos t - \sin t \end{bmatrix}$$

Solution: $c = 1 - 2i$

As the matrix is now complex, we cannot use Example 6.9; we have to use Putzer-s method. We follow Example 6.17.

First we calculate the eigenvalues of A_{1-2i}:

$$|A_{1-2i} - \mu\, I| = \begin{vmatrix} 1-\mu & 1-2i \\ -1 & -1-\mu \end{vmatrix} = (1-\mu)(-1-\mu) + 1 - 2i = \mu^2 - 1 + 1 - 2i = \mu^2 - 2i = 0.$$

For the eigenvalues, we need to calculate the square root of $2i$.

$$\begin{aligned} i &= e^{i\left(\frac{1}{2}+2k\right)\pi} \\ \sqrt{i} &= e^{i\left(\frac{1}{4}+k\right)\pi} = \pm\frac{\sqrt{2}}{2}(1+i) \\ \mu_{12} &= \sqrt{2i} = \pm(1+i) \end{aligned}$$

As we have two distinct roots, we are at Case II of Example 6.17. We compute

$$\begin{aligned} r_1(t) &= e^{\mu_1 t} = e^{-(1+i)t} \\ r_2(t) &= \frac{e^{\mu_2 t} - e^{\mu_1 t}}{\mu_2 - \mu_1} = \frac{e^{(1+i)t} - e^{-(1+i)t}}{2(1+i)}. \end{aligned}$$

Using these and formula (6.13), we get:

$$e^{tA_{1-2i}} = \frac{e^{\mu_1 t}}{\mu_1 - \mu_2}\left(A_{1-2i} - \mu_2\, I\right) + \frac{e^{\mu_2 t}}{\mu_2 - \mu_1}\left(A_{1-2i} - \mu_1\, I\right).$$

Exercise 6.24

Exercise

1. Find the 2-dimensional real vector space of solutions of the following two homogeneous equations for real-valued functions
$$\ddot{x} + x = 0 \qquad \text{and} \qquad \ddot{x} - x = 0.$$

2. In this and the following part consider the equation system studied in Example 2.16:
$$\ddot{x}_1 = x_2 \qquad \ddot{x}_2 = x_1.$$
Argue that the solutions found in Exercise 2.22 form a basis for the space of maximal solutions.

3. Using part 1, determine again all solutions of the equations system. Compare with part 2. Hint: Example 5.21.

Solution (part 1)

We begin with the first equation. We identify $\mathcal{D}_Y = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. First, we determine the associated first-order ODE:
$$\begin{aligned} y_1 &= x \\ y_2 &= \dot{x} \end{aligned} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \mathcal{D}_X = \mathbb{R} \times \mathbb{R}^2.$$

According to Theorem 6.12, the space of maximal solutions is the following:
$$S_{0,y} = \left\{ \left(\mathbb{R}, e^{tA}\eta\right) \mid \eta \in \mathbb{R}^2 \right\}.$$

As $\operatorname{Tr} A = 0$, we can use Example 6.9 to determine the exponential:
$$e^{tA} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

To find the solution space of the second-order ODE, we can apply Theorem 6.22 (extract the first row of $S_{0,y}$):
$$S_0 = \{(\mathbb{R}, \eta_1 \cos t + \eta_2 \sin t) \mid \eta_1, \eta_2 \in \mathbb{R}\}.$$

We can follow the same steps for the second equation. Identify $\mathcal{D}_Y = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The associated first-order ODE:
$$\begin{aligned} y_1 &= x \\ y_2 &= \dot{x} \end{aligned} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \mathcal{D}_X = \mathbb{R} \times \mathbb{R}^2.$$

The space of maximal solutions:
$$S_{0,y} = \left\{ \left(\mathbb{R}, e^{tA}\eta\right) \mid \eta \in \mathbb{R}^2 \right\}.$$

$\operatorname{Tr} A = 0$ again, so the exponential:
$$e^{tA} = \begin{bmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{bmatrix}$$

The solution space of the second-order ODE:
$$S_0 = \{(\mathbb{R}, \eta_1 \operatorname{ch} t + \eta_2 \operatorname{sh} t) \mid \eta_1, \eta_2 \in \mathbb{R}\}.$$

Solution (part 2)

This second-order homogeneous system of ODE-s can be rewritten in the form of formula (6.20):

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{A_0}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0.$$

Then, the associated first-order ODE is the following:

$$\begin{aligned} y_1 &= x_1 \\ y_2 &= x_2 \\ y_3 &= \dot{x}_1 \\ y_4 &= \dot{x}_2 \end{aligned} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{A}\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \qquad \dot{y} = \begin{bmatrix} 0 & I \\ -A_0 & -A_1 \end{bmatrix} y$$

To find the solution space, we would like to use Theorem 6.22, so we need to calculate e^{tA}. We can easily see by Laplace expansion with respect to the first row that the determinant is $|A| = -1$, so the matrix can be diagonalized. We can identify the eigenvalues by solving

$$\begin{vmatrix} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 0 & 1 \\ 0 & 1 & -\lambda & 0 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda\begin{vmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & -\lambda & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda^4 - 1 = 0.$$

The four eigenvalues are

$$\lambda_1 = 1 \qquad \lambda_2 = -1 \qquad \lambda_3 = i \qquad \lambda_4 = -i.$$

The corresponding eigenvectors:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \qquad v_3 = \begin{bmatrix} i \\ -i \\ -1 \\ 1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} -i \\ i \\ -1 \\ 1 \end{bmatrix}.$$

Now we can construct the matrix from the eigenvectors, and calculate its inverse:

$$C = \begin{bmatrix} 1 & -1 & i & -i \\ 1 & -1 & -i & i \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad C^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -i & i & -1 & 1 \\ i & -i & -1 & 1 \end{bmatrix}$$

Now e^{tA} can be calculated easily by applying our results from Exercise 6.7 (see also Remark 6.14):

$$e^{tA} = C\,\mathrm{Diag}\left\{e^t,\, e^{-t},\, e^{it},\, e^{-it}\right\}C^{-1} =$$

$$= \frac{1}{2}\begin{bmatrix} \operatorname{ch} t & \operatorname{ch} t & \operatorname{sh} t & \operatorname{sh} t \\ \operatorname{ch} t & \operatorname{ch} t & \operatorname{sh} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{sh} t & \operatorname{ch} t & \operatorname{ch} t \\ \operatorname{sh} t & \operatorname{sh} t & \operatorname{ch} t & \operatorname{ch} t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \cos t & -\cos t & \sin t & -\sin t \\ -\cos t & \cos t & -\sin t & \sin t \\ -\sin t & \sin t & \cos t & -\cos t \\ \sin t & -\sin t & -\cos t & \cos t \end{bmatrix}.$$

We can extract the first two rows from e^{tA} to get Ψ_1:

$$\Psi_1 = \frac{1}{2}\begin{bmatrix} \operatorname{ch} t & \operatorname{ch} t & \operatorname{sh} t & \operatorname{sh} t \\ \operatorname{ch} t & \operatorname{ch} t & \operatorname{sh} t & \operatorname{sh} t \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \cos t & -\cos t & \sin t & -\sin t \\ -\cos t & \cos t & -\sin t & \sin t \end{bmatrix}.$$

Finally, we can apply Theorem 6.22 to get the solution space.

$$S_0 = \left\{\Psi_1\eta \,|\, \eta \in \mathbb{R}^4\right\}$$

Solution (part 3)

Exercise 6.30

Exercise

Let

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Identify a particular solution y_0 and the solution space $\mathcal{S}_B$ for the following equations:

$$\begin{array}{lll} 1)\quad \dot{y} = Ay + \begin{bmatrix} 0 \\ 1 \end{bmatrix} & 3)\quad \dot{y} = Ay + \begin{bmatrix} 0 \\ e^t \end{bmatrix} & 5)\quad \dot{y} = Ay + \begin{bmatrix} 0 \\ t \end{bmatrix} \\ 2)\quad \dot{y} = By + \begin{bmatrix} 0 \\ 1 \end{bmatrix} & 4)\quad \dot{y} = By + \begin{bmatrix} 0 \\ e^t \end{bmatrix} & 6)\quad \dot{y} = By + \begin{bmatrix} 0 \\ t \end{bmatrix} \end{array}$$

Solution

According to Theorem 6.28, if $A \in \mathrm{M}_n(\mathbb{F})$ and $b \in C^0(\mathcal{I}; \mathbb{F}^n)$ where $\mathcal{I} \subseteq \mathbb{R}$ is an interval, and $t_0 \in \mathcal{I}$, the a particular solution y_0 and the solution space $\mathcal{S}_b$ of the ODE $\dot{y} = Ay + b$ is given by

$$y_0 = \int_{t_0}^{t} e^{(t-s)A}\, b(s)\ \mathrm{d}s \qquad S_b = \left\{ y_0 + e^{tA}\eta \,|\, \eta \in \mathbb{F}^n \right\}.$$

The exponentials e^{tA} and e^{tB} can be calculated according to Example 6.9, see the solution of Exercise 6.9 for the details.

$$|A| = 1 \qquad |B| = -1.$$

$$|A| > 0 \qquad e^{tA} = \cos\left(t\sqrt{|A|}\right) I + \frac{\sin\left(t\sqrt{|A|}\right)}{\sqrt{|A|}} A = \cos t\, I + \sin t\, A = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

$$|B| < 0 \qquad e^{tB} = \mathrm{ch}\left(t\sqrt{-|B|}\right) I + \frac{\mathrm{sh}\left(t\sqrt{-|B|}\right)}{\sqrt{-|B|}} B = \mathrm{ch}\, t\, I + \mathrm{sh}\, t\, B = \begin{bmatrix} \mathrm{ch}\, t & \mathrm{sh}\, t \\ \mathrm{sh}\, t & \mathrm{ch}\, t \end{bmatrix}$$

We calculate the general form of the particular solutions y_0 for a general $b(t)$:

$$y_0 = \int_{t_0}^{t} e^{(t-s)A}\, b(s)\ \mathrm{d}s = \int_{t_0}^{t} \begin{bmatrix} \cos(t-s) & \sin(t-s) \\ -\sin(t-s) & \cos(t-s) \end{bmatrix} \begin{bmatrix} b_1(s) \\ b_2(s) \end{bmatrix} \mathrm{d}s$$
$$y_0 = \int_{t_0}^{t} e^{(t-s)B}\, b(s)\ \mathrm{d}s = \int_{t_0}^{t} \begin{bmatrix} \mathrm{ch}\,(t-s) & \mathrm{sh}\,(t-s) \\ \mathrm{sh}\,(t-s) & \mathrm{ch}\,(t-s) \end{bmatrix} \begin{bmatrix} b_1(s) \\ b_2(s) \end{bmatrix} \mathrm{d}s$$

Note that we are searching for one particular solution, thus t_0 can be chosen arbitrarily. So, the solution space is then given by the formula above.

Executing the integrals, we get the following independent particular solutions:

$$\left.\begin{array}{ll} 1) & y_0(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 3) & y_0(t) = \begin{bmatrix} \frac{1}{2}e^t \\ \frac{1}{2}e^t \end{bmatrix} \\ 5) & y_0(t) = \begin{bmatrix} t \\ 1 \end{bmatrix} \end{array}\right\} \mathcal{S}_b = \left\{ y_0 + e^{tA}\,\eta \,|\, \eta \in \mathbb{R}^2 \right\} \qquad \left.\begin{array}{ll} 2) & y_0(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \\ 4) & y_0(t) = \begin{bmatrix} \frac{1}{2}\,t\,e^t \\ \frac{1}{2}\,e^t(1+t) \end{bmatrix} \\ 6) & y_0(t) = \begin{bmatrix} -t \\ -1 \end{bmatrix} \end{array}\right\} \mathcal{S}_b = \left\{ y_0 + e^{tB}\,\eta \,|\, \eta \in \mathbb{R}^2 \right\}$$

Exercise 6.31

Exercise

Identify a particular solution x_0 and the solution space $\mathcal{S}_f$ for the following ODE-s:

$$1)\quad \ddot{x}+x=1 \qquad 3)\quad \ddot{x}+x=e^t \qquad 5)\quad \ddot{x}+x=t$$

$$2)\quad \ddot{x}-x=1 \qquad 4)\quad \ddot{x}-x=e^t \qquad 6)\quad \ddot{x}-x=t$$

Solution equation 1

We begin by finding the associated first-order ODE:

$$\begin{aligned} y_1 &= x \\ y_2 &= \dot{x} \end{aligned} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This is the same equation as we solved in Exercise 6.30, which was solved by the application of Theorem 6.28. See the solution of that exercise for further details:

$$y_0(t) = \begin{bmatrix} 1-\cos t \\ \sin t \end{bmatrix} \qquad \mathcal{S}_b = \left\{ y_0 + \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \eta \,|\, \eta \in \mathbb{R}^2 \right\}.$$

As $y_1 = x$, we consider only the first row of the above solution of the first-order ODE to get the solution of the second-order ODE. Thus, a particular solution and the solution space is:

$$x_0(t) = 1 \qquad \mathcal{S}_b = \left\{ 1 + \eta_1 \cos t + \eta_2 \sin t \,|\, \eta_1,\, \eta_2 \in \mathbb{R}^2 \right\}.$$

Similarly, the other equations are corresponding to the ones in Exercise 6.30.

Exam 2009 October, Exercise 1

Exercise

Consider the following separable ODE:

$$\dot{y} = y^2 \sin(t), \tag{4}$$

defined for all t and y.

- Find the maximal solution with $y(0) = 0$
- Find the maximal solution with $y(0) = 1/3$
- Find the maximal solution with $y(0) = 1/2$ and $y(0) = 1$

Solution

To solve this exercise, we follow Section 4.1 and Section 4.2 in the book.

The equation can be written in the form

$$\dot{y} = \underbrace{y^2 \sin(t)}_{X(t,y)} \qquad \mathcal{D}_X = \mathcal{I} \times \mathcal{J}, \qquad \text{from the exercise: } \mathcal{D}_X = \mathbb{R} \times \mathbb{R}.$$

We can introduce the following notations as usual:

$$y(t_0) = \eta \qquad f(y) := y^2 \qquad q(t) := \sin(t) \qquad \text{so the equation: } \dot{y} = f(y)\, q(t).$$

As the ODE can be written in the above form, $\mathcal{I}$ and $\mathcal{J}$ are open intervals, $f \in \mathcal{C}^0(\mathcal{J})$ (as y^2 is continuous on $\mathbb{R}$) and $q \in \mathcal{C}^0(\mathcal{I})$ (as $\sin(t)$ is continuous on $\mathbb{R}$), 4 is a separable equation. The overall goal to solve the ODE using Theorem 4.6 from the book.

First identify the constant solutions (equilibria).

$$\varepsilon := \{y \in \mathbb{R} \mid f(y) = 0\} = f^{-1}(\{0\}) = \{0\}$$

So, the only constant solution is $(\mathbb{R}, 0)$, as $y(t) = 0$ is defined on the entire $\mathcal{I} = \mathbb{R}$. Recognizing that $y(t) = 0$ fulfills the initial condition $y(0) = 0$, we have solved the first part.

Now we decompose $\mathcal{J}$ to countable open domains.

$$\bigcup_i \mathcal{J}_i = \mathbb{R} \setminus \{\varepsilon\} = \underbrace{(-\infty,\, 0)}_{\mathcal{J}_-} \cup \underbrace{(0,\, +\infty)}_{\mathcal{J}_+}$$

As in $\mathcal{J}_-$ and $\mathcal{J}_+$, $f(y) \neq 0$, so it is OK to divide the ODE by $f(y)$. Thus, with the introduction of $h(y)$, we get:

$$h(y) := \frac{1}{f(y)} = \frac{1}{y^2} \qquad h(y)\, \dot{y} = q(t).$$

Following Section 4.2 from here, we calculate the below integrals:

$$Q_{t_0}(t) := \int_{t_0}^{t} q(\widetilde{t})\ \mathrm{d}\widetilde{t} = \int_{t_0}^{t} \sin(\widetilde{t})\ \mathrm{d}\widetilde{t} = \cos(t_0) - \cos(t)$$

$$H_\eta(y) := \int_{\eta}^{y} h(\widetilde{y})\ \mathrm{d}\widetilde{y} = \int_{\eta}^{y} \frac{1}{\widetilde{y}^2}\ \mathrm{d}\widetilde{y} = \frac{1}{\eta} - \frac{1}{y}$$

We can use Theorem 4.6, and determine the non-equilibrium solutions $(I,\, y)$. We see that all the η initial values (apart from the solved case when $\eta = 0$) lie in $\mathcal{J}_+$, so we deal with this only. From the theorem we know that

$$I = Q_{t_0}^{-1}\left(H_\eta(\mathcal{J}_+)\right) = Q_{t_0}\left(\left\{\frac{1}{\eta} - \frac{1}{y} \mid y \in (0,\, +\infty)\right\}\right) = Q_{t_0}\left(\left(-\infty,\, \frac{1}{\eta}\right)\right) =$$

$$= \left\{t \in \mathbb{R} \mid\ -\infty < \underbrace{1 - \cos(t)}_{Q_0(t)} < \frac{1}{\eta}\right\} = \left\{t \in \mathbb{R} \mid 1 - \frac{1}{\eta} < \cos(t)\right\},$$

where in the first step of the second line we used that $t_0 = 0$ for all parts of the exercise, so $Q_{t_0}(t) = Q_0(t)$. In order to get the solution, we need to calculate:

$$y(t) = H_\eta^{-1}\left(Q_{t_0}(t)\right) = \frac{1}{\frac{1}{\eta} - Q_0(t)} = \frac{1}{\frac{1}{\eta} - 1 + \cos(t)}.$$

Let us consider the three cases when $\eta \neq 0$ one by one.

- If $\eta = 1/3$, then

$$I = \{t \in \mathbb{R} \mid -2 < \cos(t)\} = \mathbb{R}.$$

  As this consists of a single connected component containing $t_0 = 0$, the solution fulfilling the initial value problem is

$$\left(\mathbb{R}, \frac{1}{2 + \cos(t)}\right).$$

- If $\eta = 1/2$, then

$$I = \{t \in \mathbb{R} \mid -1 < \cos(t)\} = \mathbb{R} \setminus \bigcup_{k\in\mathbb{Z}} \{(2k+1)\pi)\} = \bigcup_{k\in\mathbb{Z}} ((2k-1)\pi,\, (2k+1)\pi).$$

  This is the union of intervals, and as $t_0 = 0$ is contained in the interval where $k = 0$, the solution fulfilling the initial value problem is

$$\left((-\pi,\, +\pi),\, \frac{1}{1 + \cos(t)}\right).$$

- If $\eta = 1$, then

$$I = \{t \in \mathbb{R} \mid 0 < \cos(t)\} = \bigcup_{k\in\mathbb{Z}} \left(\left(2k - \frac{1}{2}\right)\pi,\, \left(2k + \frac{1}{2}\right)\pi\right).$$

  This is again the union of intervals, and as $t_0 = 0$ is contained again in the interval where $k = 0$, the solution fulfilling the initial value problem is

$$\left(\left(-\frac{\pi}{2}, +\frac{\pi}{2}\right),\, \frac{1}{\cos(t)}\right).$$

Please find the interactive graph of the solution by clicking on this link.

Exam 2014 August, Exercise 1

Exercise

- Show that $\begin{bmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{bmatrix}$ is a fundamental matrix for

$$\dot{y} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} y \qquad y \in \mathbb{R}^2.$$

- Find all maximal solutions for

$$\dot{y}_1 = y_1 \qquad \dot{y}_2 = y_3 \qquad \dot{y}_3 = y_2 \qquad \dot{y}_4 = y_4 \qquad\qquad y_i \in \mathbb{R}.$$

- Find all maximal solutions for $\dot{\Phi} = \Phi^T$, where $\Phi \in \mathrm{M}_2(\mathbb{R})$.

Solution (part 1)

We can write the equation of the form

$$\dot{y} = A\,y \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

According to Theorem 6., if $A \in \mathrm{M}_n(\mathbb{F})$, $\dot{y} = Ay$, then the fundamental matrix of the equation is $(\mathbb{R}, e^{tA})$. To calculate e^{tA}, we can follow Example 6.9 as $\mathrm{Tr}A = 0$.

$$|A| = -1 < 0 \qquad e^{tA} = \operatorname{ch}(t\sqrt{-|A|})\,I + \frac{\operatorname{sh}(t\sqrt{-|A|})}{\sqrt{-|A|}}\,A = \operatorname{ch}(t)\,I + \operatorname{sh}(t)\,A = \begin{bmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{bmatrix}$$

Thus, we have showed that the fundamental matrix is as stated.

Solution (part 2)

As y_1, y_4 are detached from y_2, y_3, the first and last equations can be solved separately. Following Example 1.3, the sets of maximal solutions are:

$$\left\{ \left(\mathbb{R},\, y_1 = \eta_1\, e^t\right) \mid \eta_1 \in \mathbb{R} \right\} \qquad \left\{ \left(\mathbb{R},\, y_4 = \eta_4\, e^t\right) \mid \eta_4 \in \mathbb{R} \right\}.$$

For y_2, y_3, we have the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_2 \\ y_3 \end{bmatrix}.$$

This is the same ODE as in part 1. All maximal solutions are given as the linear combinations of the columns of the fundamental matrix (Theorem 6.12). Thus, the maximal solutions of the equations are:

$$\left.\begin{array}{ll} y_1 : & (\mathbb{R},\, \eta_1\, e^t) \\ y_2 : & (\mathbb{R},\, \eta_2\, \operatorname{ch} t + \eta_3\, \operatorname{sh} t) \\ y_3 : & (\mathbb{R},\, \eta_2\, \operatorname{sh} t + \eta_3\, \operatorname{sh} t) \\ y_4 : & (\mathbb{R},\, \eta_4\, e^t) \end{array}\right\} \qquad \eta_i \in \mathbb{R}$$

Solution (part 3)

Let us denote the elements of the matrix Φ by:

$$\Phi(t) = \begin{bmatrix} y_1(t) & y_3(t) \\ y_2(t) & y_4(t) \end{bmatrix} \qquad\qquad \Phi^T(t) = \begin{bmatrix} y_1(t) & y_2(t) \\ y_3(t) & y_4(t) \end{bmatrix}.$$

Then the ODE is:

$$\begin{bmatrix} \dot{y}_1(t) & \dot{y}_3(t) \\ \dot{y}_2(t) & \dot{y}_4(t) \end{bmatrix} = \begin{bmatrix} y_1(t) & y_2(t) \\ y_3(t) & y_4(t) \end{bmatrix}.$$

This means, that we have the same set of ODE-s as in part 2 of this exercise for the elements. Thus, the set of maximal solutions is:

$$\left\{ \left(\mathbb{R}, \begin{bmatrix} \eta_1\, e^t & \eta_2\ \mathrm{sh}\, t + \eta_3\ \mathrm{ch}\, t \\ \eta_2\ \mathrm{ch}\, t + \eta_3\ \mathrm{sh}\, t & \eta_4\, e^t \end{bmatrix} \right) \mid \eta_i \in \mathbb{R} \right\}$$

Exam 2014 August, Exercise 2

Exercise

Consider the linear fourth-order ODE:

$$x^{(4)} - 5x^{(2)} + 4x = 0, \qquad x \in \mathbb{R}.$$

We can use that

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 4 & 1 & 4 \\ 1 & 8 & -1 & -8 \end{bmatrix}^{-1} = \frac{1}{12} \begin{bmatrix} 8 & 8 & -2 & -2 \\ -2 & -1 & 2 & 1 \\ 8 & -8 & -2 & 2 \\ -2 & 1 & 2 & -1 \end{bmatrix}.$$

1. Find the space of maximal solutions S_0.
2. Find the maximal solution fulfilling

$$x(0) = 1, \qquad x^{(1)}(0) = 1, \qquad x^{(2)}(0) = -5, \qquad x^{(3)}(0) = 1.$$

3. Find $\Psi_{14}(t)$, the top right corner of e^{tA}, where A is the coefficient matrix for the associated first-order ODE $\dot{y} = Ay$.
4. Find the maximal solution of

$$x^{(4)} - 5x^{(2)} + 4x = 12e^t,$$

that satisfies the above IVP.

Solution (part 1)

First, compute the associated first-order ODE:

$$\begin{aligned} y_1 &= x^{(0)} \\ y_2 &= x^{(1)} \\ y_3 &= x^{(2)} \\ y_4 &= x^{(3)} \end{aligned} \qquad \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & 0 & 5 & 0 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

To find the solution space, we would like to use Theorem 6.22, so we need to calculate e^{tA}. We can easily see by Laplace expansion with respect to the first row that the determinant is $|A| = 4$, so the matrix can be diagonalized. We can identify the eigenvalues by solving $|A - \lambda I| = 0$, or by finding the characteristic polynomial from the higher-order equation (by substituting $e^{\lambda t}$):

$$\lambda^4 - 5\lambda^2 + 4 = 0.$$

The four eigenvalues are

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = -1 \qquad \lambda_4 = -2.$$

The solution space is then, by Theorem 6.23:

$$S_0 = \mathrm{Span}_{\mathbb{R}} \left\{ e^t,\, e^{2t},\, e^{-t},\, e^{-2t} \right\}.$$

Solution (part 2)

To find the maximal solution fulfilling the IVP, we take a general element of S_0:

$$x_0(t) = \eta_1 e^t + \eta_2 e^{2t} + \eta_3 e^{-t} + \eta_4 e^{-2t}.$$

Considering the derivatives at $t = 0$, we have:

$$\begin{bmatrix} x^{(0)}(0) \\ x^{(1)}(0) \\ x^{(2)}(0) \\ x^{(3)}(0) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 4 & 1 & 4 \\ 1 & 8 & -1 & -8 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -5 \\ 1 \end{bmatrix}.$$

Solving this system of equation by multiplying the the inverse with the initial conditions will result in:

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

Solution (part 3)

Now we calculate the eigenvectors of A:

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 8 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ -2 \\ 4 \\ -8 \end{bmatrix}.$$

Now we can construct the matrix C from the eigenvectors, and calculate its inverse using the hint:

$$C = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 4 & 1 & 4 \\ 1 & 8 & -1 & -8 \end{bmatrix} \qquad C^{-1} = \frac{1}{12} \begin{bmatrix} 8 & 8 & -2 & -2 \\ -2 & -1 & 2 & 1 \\ 8 & -8 & -2 & 2 \\ -2 & 1 & 2 & -1 \end{bmatrix}$$

Now e^{tA} can be calculated easily by applying our results from Exercise 6.7 (see also Remark 6.14):

$$e^{tA} = C \, \mathrm{Diag} \left\{ e^t, \, e^{2t}, \, e^{-t}, \, e^{-2t} \right\} C^{-1}$$

We can extract Ψ_{14}:

$$\Psi_{14} = \frac{1}{12} \left(-2e^t + e^{2t} + 2e^{-t} - e^{-2t} \right) = \frac{1}{6} \operatorname{sh} 2t - \frac{1}{3} \operatorname{sh} t.$$

Solution (part 4)

To get the solution for the inhomogeneous equation, we apply Theorem 6.30.

$$x_0(t) = \int_{t_0}^{t} \Psi_{14}(t - s) \, f(s) \, \mathrm{d}s \qquad f(t) = 12e^t$$

Executing the calculation we get:

$$x_0(t) = \cdots = -2te^t.$$

So, the maximal solutions to the inhomogeneous equation are:

$$S_b = S_0 - 2te^t.$$

In order to find the coefficients so that the solution satisfies the IVP, we can use a similar method as above in part 2.

$$\begin{bmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 20 \\ 0 \\ 0 \\ -8 \end{bmatrix}.$$

The maximal solution satisfying the IVP is then

$$\left(\mathbb{R}, \, \left(\frac{5}{3} - 2t \right) e^t - \frac{2}{3} e^{-2t} \right).$$

Exam 2016 March, Exercise 1

Exercise

Consider the equations defined in $\mathcal{D}_X = \mathbb{R}_+ \times \mathbb{R}_+$, and the solutions $y = y(t)$.

1. Find the maximal solutions for
$$\dot{y} = \frac{1}{4y^3}, \qquad y(2) = \sqrt{2}.$$
2. Find the maximal solutions for
$$\dot{y} = \frac{2t}{4y^3}, \qquad y(\sqrt{2}) = \sqrt{2}.$$
Hint: reparametrization (temporal transformation or separation of variables).
3. Find the maximal solutions for
$$\dot{y} = -\frac{y}{4t} + \frac{1}{2y^3}, \qquad y(\sqrt{2}) = 2^{\frac{3}{8}}.$$
Hint: search for a transformation of the form $t^\alpha\, y(t)$.

Solution (part 1)

This ODE can be solved by the separation of variables as it has the form of $\dot{y} = f(y)\, q(t)$. We can identify the two components of $\mathcal{D}_X$:
$$\mathcal{D}_X = \mathcal{I} \times \mathcal{J} = \mathbb{R}_+ \times \mathbb{R}_+.$$
Furthermore,
$$f(y) := \frac{1}{4y^3} \qquad q(t) := 1 \qquad t_0 = 2 \qquad \eta = \sqrt{2}$$
First, identify the constant solutions:
$$\mathcal{E} = \{y \in \mathcal{J} | f(y) = 0\} = \left\{ y \in \mathbb{R}_+ \mid \frac{1}{4y^3} = 0 \right\} = \emptyset.$$
Thus, there are no constant solutions, so there is no need to decompose $\mathcal{J}$ further. Define $h(y)$:
$$h(y) := \frac{1}{f(y)} = 4y^3,$$
so the ODE now has the form of
$$h(y)\dot{y} = q(t).$$
To integrate the separated ODE, we calculate the antiderivatives of $q(t)$ and $h(y)$:
$$Q_{t_0}(t) = \int_{t_0}^{t} q(s)\,\mathrm{d}s = \int_{t_0}^{t} 1\,\mathrm{d}s = t - t_0 = t - 2,$$
$$H_\eta(y) = \int_{\eta}^{y} h(r)\,\mathrm{d}r = \int_{\eta}^{y} 4r^3\,\mathrm{d}r = y^4 - \eta^4 = y^4 - 4.$$

According to Theorem 4.6, the interval of the maximal solution is the connected component of t_0 in $Q_{t_0}^{-1}(H_\eta(\mathcal{J}))$, and the function solving the ODE is given by $y(t) = H_\eta^{-1}(Q_{t_0}(t))$. So we compute first $H_\eta(\mathcal{J})$ and then I.
$$H_\eta(\mathcal{J}) = \left\{ z \in \mathbb{R} | 0^4 - 4 < z < \infty^4 - 4 \right\} = (-4, \infty)$$
$$I = Q_{t_0}^{-1}(H_\eta(\mathcal{J})) = (-4 + 2, \infty + 2) = (-2, \infty)$$

But as $\mathcal{I} = \mathbb{R}_+$, $I = \mathbb{R}_+$ as well. We move on to compute H_η^{-1} and then $y(t)$.

$$H_\eta^{-1} = (z+4)^{\frac{1}{4}}$$
$$y(t) = H_\eta^{-1}(Q_{t_0}(t)) = (t-2+4)^{\frac{1}{4}} = (t+2)^{\frac{1}{4}}$$

So, the maximal solution satisfying the IVP is

$$\left(\mathbb{R}_+, (t+2)^{\frac{1}{4}}\right)$$

Solution (part 2)

We solve this ODE by the application of a temporal transformation, and begin by applying formula (5.3).

$$\mathcal{D}_{X_\Theta} = \{(t,y) \in \mathcal{I}_\Theta \times \mathbb{R}^n \mid (\Theta(t), y) \in \mathcal{D}_X\}$$
$$X_\Theta(t,y) = \dot{\Theta}(t)\, X\left(\Theta(t), y\left(\Theta(t)\right)\right)$$

Substituting $X(t,y) = \frac{2t}{4y^3}$ into the above formula, we get

$$X_\Theta(t,y) = \dot{\Theta}(t)\,\frac{2\Theta(t)}{4y_\Theta^3}.$$

We could get back the original equation if $2\,\dot{\Theta}(t)\,\Theta(t)$ was 1.

$$2\,\dot{\Theta}(t)\,\Theta(t) = 1 \qquad \Longrightarrow \qquad \Theta(t) = \sqrt{t}$$

This temporal transformation defined on $\mathcal{I}_\Theta = \mathbb{R}_+$, is differentiable, invertible and orientation preserving (as $\dot{\Theta} > 0$). We identify $\mathcal{D}_{X_\Theta}$ based on the formula above:

$$\mathcal{D}_{X_\Theta} = \left\{(t,y) \in \mathbb{R}_+ \times \mathbb{R}_+ \mid (\sqrt{t}, y) \in \mathcal{D}_X\right\} = \mathcal{D}_X.$$

Now we transform the IVP of the original equation to the transformed equation. Values in the transformed time are those that were $\Theta^{-1}(t)$ earlier in the original time:

$$y(\sqrt{2}) = \sqrt{2} \qquad \Longrightarrow \qquad y(2) = \sqrt{2}.$$

The solution to the transformed equation is the same as above as all the equation, the IVP and the domain are the same.

$$\left(\mathbb{R}_+, (t+2)^{\frac{1}{4}}\right)$$

To transform back to the original time, we apply the $t \to \Theta^{-1}(t) = t^2$ inverse transformation, so the solution of the original equation is

$$\left(\mathbb{R}_+, (t^2+2)^{\frac{1}{4}}\right).$$

Note that the interval $\mathbb{R}_+$ is left intact by the inverse transformation as well. This solution is maximal due to Theorem 5.9.

Solution (part 3)

We solve this equation using linear spatial transformation. According to Definition 5.12, the relationship between the original and the transformed solutions is

$$y(t) = \Phi(t) y_\psi(t) + g(t), \tag{5}$$

where $\Phi(t)$ in our 1-dimensional case is a scalar valued function. Using the hint, we know that that $\Phi(t) = t^\alpha$ and $g(t) = 0$. We know that $X(t,y) = -y/(4t) + 1/(2y^3)$, and to get $X_\psi(t, y_\psi)$, we can use formula (5.11) from the book (which is derived by substituting the above expression into the original ODE):

$$X_\psi(t, y_\psi) = \Phi^{-1}\left(-\dot{\Phi} y_\psi - \dot{g} + X\left(t, \Phi y_\psi + g\right)\right) = t^{-\alpha}\left(-\alpha\, t^{\alpha-1} y_\psi - \frac{t^\alpha y_\psi}{4t} + \frac{1}{2(t^\alpha y_\psi)^3}\right)$$

where we neglected the t arguments for simplicity. Expanding the expression results in:

$$X_\psi(t, y_\psi) = -\left(\alpha + \frac{1}{4}\right)\frac{y}{t} + \frac{1}{2\,t^{4\alpha}\,y^3},$$

from which we can guess that $\alpha = -1/4$ result in the simplification of the ODE. Defined on $\mathcal{I}_\psi = \mathbb{R}_+$, the linear spatial transformation we can apply is

$$\left(\mathbb{R}_+,\, t^{-\frac{1}{4}}\right),$$

which fulfills all criteria of Definition 5.12 of linear spatial transformation, as $\Phi(t) = t^{-\frac{1}{4}}$ is invertible on $\mathbb{R}_+$. The transformed equation is the same as the ODE in part 2:

$$\dot{y}_\psi = X_\psi(t, y_\psi) \qquad X_\psi(t, y_\psi) = \frac{2t}{4y_\psi^3}$$

Now we use formula 5.12 to define the region.

$$\mathcal{D}_{X_\psi} = \{(t,\, y_\psi) \in \mathcal{I}_\psi \times \mathbb{F}^n \,|\, (t, \Phi y_\psi + g) \in \mathcal{D}_X\} = \left\{(t,\, y_\psi) \in \mathbb{R}_+ \times \mathbb{R} \,|\, (t, t^{-\frac{1}{4}} y_\psi) \in \mathbb{R}_+ \times \mathbb{R}\right\} = \mathcal{D}_X$$

The maximal solution of the transformed equation is

$$\left(\mathbb{R}_+, (t^2 + 2)^{\frac{1}{4}}\right),$$

and to get the solution to the original ODE, we apply the transformation to this solution:

$$\left(\mathbb{R}_+, t^{-\frac{1}{4}}(t^2 + 2)^{\frac{1}{4}}\right).$$

This solution is maximal due to Theorem 5.13. The domain of the solution is left intact as according to formula (5.14),

$$\mathcal{D}_{(X_\psi)_{\psi^{-1}}} = (\mathcal{I}_\psi \times \mathbb{F}^n) \cap \mathcal{D}_X = (\mathbb{R}_+ \times \mathbb{R}) \cap \mathcal{D}_X = D_X.$$

At $t = \sqrt{2}$, the above solution takes the value of $2^{\frac{3}{8}}$, so the IVP is fulfilled.

Exam 2017 March, Exercise 3

Exercise

1. Find the maximal solutions for the homogeneous equation
$$\ddot{x} - \dot{x} - 2x = 0 \qquad \forall t \in \mathbb{R}.$$
2. Find a particular solution for the inhomogeneous equation
$$\ddot{x} - \dot{x} - 2x = -3e^{-t}$$
3. Argue that the inhomogeneous equation with initial condition $x(0) = 0$, $\dot{x}(0) = 1$ has a unique solution, and find it.

Solution (part 1, version 1)

First find the associated first-order ODE:

$$\begin{aligned} y_1 &= x \\ y_2 &= \dot{x} \end{aligned} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

We know from Theorem 6.12 that the space of maximal solutions is

$$S_0 = \left\{ \left(\mathbb{R},\, e^{tA}\eta\right) \mid \eta \in \mathbb{R}^2 \right\}.$$

As $\operatorname{Tr} A = 1 \neq 0$, we can use Remark 6.10 to decompose the matrix to the sum where one of the terms has trace 0, and the other is diagonal:

$$A = \underbrace{A - \frac{1}{2}I}_{A_1,\ \operatorname{Tr}(A_1)=0} + \underbrace{\frac{1}{2}I}_{A_2,\ \text{diagonal matrix}} = \begin{bmatrix} -1/2 & 1 \\ 2 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}.$$

Then, by Proposition 6.1,

$$e^{tA} = e^{t(A_1 + A_2)} = e^{t\,A_1} \cdot e^{t\,A_2}.$$

The first term, e^{A_1} can be calculated from Example 6.9:

$$|A_1| = -9/4 < 0 \qquad \begin{aligned} e^{tA_1} &= \operatorname{ch}\left(t\sqrt{-|A_1|}\right) I + \frac{\operatorname{sh}\left(t\sqrt{-|A_1|}\right)}{\sqrt{-|A_1|}} A_1 = \\ &= \operatorname{ch}\left(\frac{3}{2}t\right) I + \operatorname{sh}\left(\frac{3}{2}t\right) A_1 = \\ &= \begin{bmatrix} \operatorname{ch}\left(\frac{3}{2}t\right) - \frac{1}{3}\operatorname{sh}\left(\frac{3}{2}t\right) & \frac{2}{3}\operatorname{sh}\left(\frac{3}{2}t\right) \\ \frac{4}{3}\operatorname{sh}\left(\frac{3}{2}t\right) & \operatorname{ch}\left(\frac{3}{2}t\right) + \frac{1}{3}\operatorname{sh}\left(\frac{3}{2}t\right) \end{bmatrix}. \end{aligned}$$

The second term, e^{A_2} can be calculated by Example 6.4:

$$e^{tA_2} = e^{\frac{1}{2}t}\, I.$$

Thus, by multiplying the terms, we get:

$$\begin{aligned} e^{tA} &= e^{tA_1} \cdot e^{tA_2} = \\ &= e^{\frac{1}{2}t} \begin{bmatrix} \operatorname{ch}\left(\frac{3}{2}t\right) - \frac{1}{3}\operatorname{sh}\left(\frac{3}{2}t\right) & \frac{2}{3}\operatorname{sh}\left(\frac{3}{2}t\right) \\ \frac{4}{3}\operatorname{sh}\left(\frac{3}{2}t\right) & \operatorname{ch}\left(\frac{3}{2}t\right) + \frac{1}{3}\operatorname{sh}\left(\frac{3}{2}t\right) \end{bmatrix} = \\ &= \frac{e^{2t}}{3}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} + \frac{e^{-t}}{3}\begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}. \end{aligned}$$

According to Theorem 6.12, the maximal solutions are then:

$$\left\{ \left(\mathbb{R},\, e^{tA}\eta\right) \mid \eta \in \mathbb{R}^2 \right\}.$$

As the first row of the solution, y_1 corresponds to the original solution x (see Theorem 6.22), so the set of maximal solutions for the original equation is:

$$\left\{ \left(\mathbb{R},\, \eta_1\, e^{2t} + \eta_2\, e^{-t}\right) \mid \eta_1, \eta_2 \in \mathbb{R}^2 \right\},$$

where we simplified the coefficients.

Solution (part 1, version 2)

We can use Theorem 6.23 from the book. We have a 1-dimensional, second-order ODE, and in our case $\mathbb{F} = \mathbb{R}$. We can obtain the characteristic polynomial by substituting $x(t) = e^{\lambda\, t}$ to the original ODE (see Lemma 6.25):

$$\lambda^2 - \lambda - 2 = 0 \qquad \Longrightarrow \qquad \lambda_1 = 2 \qquad \lambda_2 = -1.$$

As both eigenvalues have a multiplicity of 1, by Theorem 6.23, solution space is:

$$\mathcal{S}_0 = \mathrm{Span}_{\mathbb{R}} \left\{ e^{2t},\, e^{-t} \right\}.$$

Solution (part 2)

We would like to use Theorem 6.30, which states that if $A_j \in \mathrm{M}_n(\mathbb{F})$, and $f \in C^0(\mathcal{I}; \mathbb{F}^n)$ with $\mathcal{I} \subseteq \mathbb{R}$ an interval, then a particular solution of the higher-order ODE

$$x^{(k)} + A_{k-1} x^{(k-1)} + \cdots + A_1 x^{(1)} + A_0 x = f(t)$$

is given by

$$x_0(t) = \int_{t_0}^{t} \Psi_{1k}(t-s)\, f(s)\, \mathrm{d}s,$$

and the solution space is

$$\mathcal{S}_f = \left\{ \int_{t_0}^{t} \Psi_{1k}(t-s)\, f(s)\, \mathrm{d}s + \Psi_1(t)\, \eta \,\middle|\, \eta \in \mathbb{F}^{kn} \right\}.$$

In our case (Theorem 6.30),

$$\Psi_{1k} = \Psi_{12} = \frac{1}{3}\left(e^{2t} - e^{-t}\right).$$

So, substituting this into the formula above:

$$x_0 = \int_0^t \frac{1}{3}\left(e^{2t} - e^{-t}\right)\left(-3e^{-s}\right)\, \mathrm{d}s = \cdots = \frac{1}{3}\left(e^{2t} - e^{-t}\right) + t\, e^{-t}.$$

As the first term of x_0 is included in the solution of the homogeneous equation, it can be neglected for simplicity. Now we can determine the solution space of maximal solutions for the inhomogeneous ODE.

$$\left\{ \left(\mathbb{R},\, \eta_1\, e^{2t} + \eta_2\, e^{-t} + t\, e^{-t}\right) \mid \eta_1, \eta_2 \in \mathbb{R}^2 \right\}$$

Solution (part 3)

The associated first-order ODE of the inhomogeneous ODE looks like:

$$\begin{aligned} y_1 &= x \\ y_2 &= \dot{x} \end{aligned} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = X(t,\, y) = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3e^{-t} \end{bmatrix}.$$

We would like to use Theorem 7.19. Assume that the solution to the IVP is not unique. There exists a time $t_0 = 0$, where the two solutions must match. $X(t,\, y)$ is locally Lipschitz, as it has continuous partial y-derivatives (Lemma 7.14). It follows from Theorem 7.19 that the two solutions must be the same.

To satisfy the IVP, $\eta_1 = \eta_2 = 0$, so the maximal solution satisfying the IVP is:

$$\left(\mathbb{R},\, t\, e^{-t}\right).$$

Exam 2018 June, Exercise 2

Exercise

1. Find the fundamental matrix of
$$\dot{y} = By \qquad B = \begin{bmatrix} 2 & -1 \\ 5 & -2 \end{bmatrix}.$$

2. Find the maximal solutions of the equation.

3. Find the maximal solution satisfying the IVP
$$y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

4. Find the maximal solution for the inhomogeneous equation
$$\dot{y} = By + \begin{bmatrix} t-2 \\ 2t-4 \end{bmatrix} \qquad y(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solution (part 1)

We know from Theorem 6.12 that the space of solutions is

$$S_0 = \left\{ \left(\mathbb{R},\, e^{tB}\eta\right) | \eta \in \mathbb{R}^2 \right\}.$$

As $\operatorname{Tr} B = 0$, we can use Example 6.9 to calculate the exponential ($|B| = 1 > 0$).

$$e^{tB} = \begin{bmatrix} \cos t + 2\sin t & -\sin t \\ 5\sin t & \cos t - 2\sin t \end{bmatrix}.$$

From Theorem 6.13, the fundamental matrix Φ is:

$$\Phi(t) = e^{tB}\eta \qquad \eta \in \mathbb{R}^2.$$

Solution (part 2)

The maximal solutions of the equation are (Theorem 6.13):

$$\left\{ (\mathbb{R},\, \Phi(t)) \mid \eta \in \mathbb{R}^2 \right\}.$$

Solution (part 3)

Again, according to Theorem 6.13, the $\eta = \Phi(0) = (1,\, 0)$, meaning that the maximal solution satisfying the IVP is

$$\left(\mathbb{R},\, \begin{bmatrix} \cos t + 2\sin t \\ 5\sin t \end{bmatrix} \right).$$

Solution (part 4)

To find a particular solution y_0, we can use Theorem 6.28.

$$y_0 = \int_{t_0}^{t} e^{(t-s)B}\, b(s)\,\mathrm{d}s = \int_{0}^{t} e^{(t-s)B}\, b(s)\,\mathrm{d}s$$

Another, faster method is to look at the inhomogeneous term, and search a particular solution in the same form

$$y_0 = \begin{bmatrix} at + b \\ ct + d \end{bmatrix}.$$

Substituting this into the equation, we get

$$\begin{bmatrix} a \\ c \end{bmatrix} = \begin{bmatrix} 2at + 2b - ct - d + t - 2 \\ 5at + 5b - 2cc - -2d + 2t - 4 \end{bmatrix}.$$

Matching the linear and constant coefficients, we get the following equation:

$$\begin{bmatrix} 2 & 0 & -1 & 0 \\ -1 & 2 & 0 & -1 \\ 5 & 0 & -2 & 0 \\ 0 & 5 & -1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \\ 4 \end{bmatrix}.$$

Solving this e.g. by Gaussian elimination, the solutions are $a = d = 0$, $b = c = 1$. Thus, a particular solution is:

$$y_0(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}.$$

Thus, the solution space for the inhomogeneous equation is (Theorem 6.28):

$$S_b = \left\{ \left(\mathbb{R},\, e^{tB}\eta + y_0\right) | \eta \in \mathbb{R}^2 \right\}.$$

To satisfy the IVP, $\eta_1 = \eta_2 = 0$, so the maximal solution satisfying the IVP is:

$$\left(\mathbb{R}, \begin{bmatrix} 1 \\ t \end{bmatrix}\right).$$

Exam 2022 August, Exercise 2

Exercise

1. Find all maximal solutions to the following system of differential equations

$$\dot{y}_1 = 2y_1y_2$$
$$\dot{y}_2 = 1$$

   defined for all t, y_1, $y_2 \in \mathbb{R}$.

2. Consider now the non-linear differential equation

$$\dot{y} = X(y) = \begin{bmatrix} \frac{1}{2}(y_1^2 - y_2^2) + 1 \\ \frac{1}{2}(y_2^2 - y_1^2) + 1 \end{bmatrix}$$

   defined for $t \in \mathbb{R}$ and $y \in \mathbb{R}^2$, i.e., $\mathcal{D}_X = \mathbb{R} \times \mathbb{R}^2$. Verify that

$$\Psi_t(y) = \begin{bmatrix} y_1 + y_2 \\ -y_1 + y_2 \end{bmatrix}$$

   with $\mathcal{I}_\Psi = \mathbb{R}$ defines a linear spatial transformation.

3. Determine the transformed region $\mathcal{D}_{X_\Psi} \subseteq \mathbb{R} \times \mathbb{R}^2$ and the the ODE $\dot{y}_\Psi = X_\Psi(t, y_\Psi)$.

4. Find the maximal solution (I, y) which satisfies

$$y(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Solution (part 1)

The second equation is independent of y_1, and can be solved using the fundamental theorem of calculus:

$$y_2(t) = \int_{t_2}^{t} 1 \,\mathrm{d}s = t - t_2 = t + \eta_2,$$

where we introduced a new variable η_2 for consistency. Then, by substituting y_2 into the first equation:

$$\dot{y}_1 = 2(t + \eta_2)y_1,$$

which can be solved as Example 1.3:

$$\left\{(\mathbb{R},\, e^G\,\eta_1) \mid \eta_1 \in \mathbb{R}\right\}$$
$$G = \int 2(t + \eta_2)\,\mathrm{d}t = t^2 + 2\eta_2\,t.$$

So, the maximal solutions for y_1 are

$$\left\{(\mathbb{R},\, e^{t^2 + 2\eta_2\,t}\,\eta_1) \mid \eta_1, \eta_2 \in \mathbb{R}\right\}.$$

Thus, the maximal solutions for the system of ODE-s:

$$\left\{\left(\mathbb{R},\, \begin{bmatrix} e^{t^2 + 2\eta_2\,t}\,\eta_1 \\ t + \eta_2 \end{bmatrix}\right) \mid \eta_1, \eta_2 \in \mathbb{R}\right\}.$$

Solution (part 2)

We can rewrite $\Psi_t(y)$ as follows:

$$\Psi_t(y) = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{\Phi(t)} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

As $\Phi(t)$ is a matrix independent of y. Its determinant is $|\Phi(t)| = 2$, so it is invertible, and the domain $\mathcal{I}_\Psi = \mathbb{R}$ on which it is given is given is an interval. Thus, all conditions of Definition 5.12 are fulfilled, and $\Psi_t(y)$ is a linear spatial transformation.

Solution (part 3)

To get $X_\psi(t, y_\psi)$, we can use formula (5.11) from the book (which is derived by substituting the above expression into the original ODE):

$$X_\psi(t, y_\psi) = \Phi^{-1} \left(-\dot{\Phi} y_\psi - \dot{g} + X\left(t, \Phi y_\psi + g\right) \right).$$

To compute the above expression, we need to calculate Φ^{-1}:

$$\Phi^{-1} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix},$$

and also note that $\dot{\Phi} = 0$. Thus,

$$X_\psi(t, y_\psi) = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2}\left((y_{\Psi 1} + y_{\Psi 2})^2 - (-y_{\Psi 1} + y_{\Psi 2})^2\right) + 1 \\ \frac{1}{2}\left((-y_{\Psi 1} + y_{\Psi 2})^2 - (y_{\Psi 1} + y_{\Psi 2})^2\right) + 1 \end{bmatrix} = \begin{bmatrix} 2y_{\Psi 1} y_{\Psi 2} + 1 \\ 1 \end{bmatrix}.$$

Now we use formula 5.12 to define the region.

$$\mathcal{D}_{X_\psi} = \{(t,\, y_\psi) \in \mathcal{I}_\psi \times \mathbb{F}^n \,|\, (t, \Phi y_\psi + g) \in \mathcal{D}_X\} = \left\{(t,\, y_\psi) \in \mathbb{R} \times \mathbb{R}^2 \,|\, (t, t y_\psi) \in \mathbb{R} \times \mathbb{R}^2\right\} = \mathcal{D}_X$$

Solution (part 4)

As the transformed ODE is the same as the one in part 1, the maximal solutions for y_Ψ are also the same:

$$\left\{ \left(\mathbb{R}, \begin{bmatrix} e^{t^2 + 2\eta_2 t}\, \eta_1 \\ t + \eta_2 \end{bmatrix} \right) |\, \eta_1, \eta_2 \in \mathbb{R} \right\}.$$

We apply the transformation to get the maximal solution for the original ODE:

$$\left\{ \left(\mathbb{R}, \begin{bmatrix} e^{t^2 + 2\eta_2 t}\, \eta_1 + t + \eta_2 \\ -e^{t^2 + 2\eta_2 t}\, \eta_1 + t + \eta_2 \end{bmatrix} \right) |\, \eta_1, \eta_2 \in \mathbb{R} \right\}.$$

Note that the domain was not affected by the transformation. The IVP can be solved easily:

$$\eta_1 + \eta_2 = 1$$
$$-\eta_1 + \eta_2 = -1,$$

which is solved by $\eta_1 = 2$ and $\eta_2 = -1$. So the maximal solution satisfying the IVP is:

$$\left(\mathbb{R}, \begin{bmatrix} 2\, e^{t^2 - 2t} + t - 1 \\ -2\, e^{t^2 - 2t} + t - 1 \end{bmatrix} \right).$$