Stochastic Models of Higher-Order Networks

Point Processes and Topological Data Analysis

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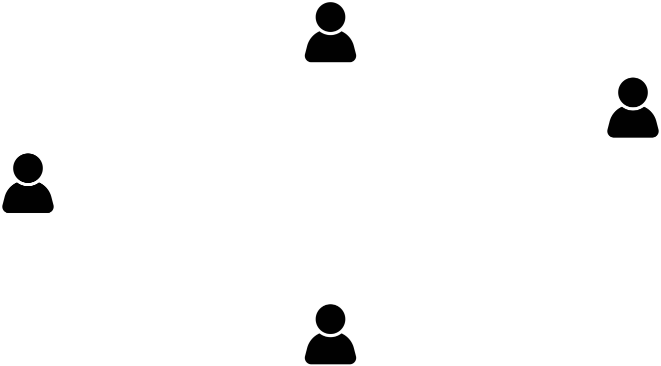
Complex Systems



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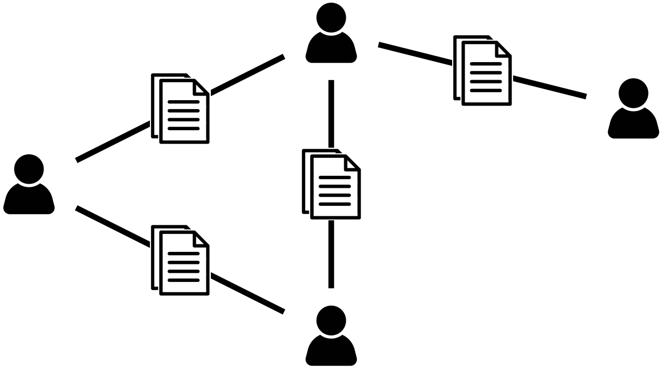

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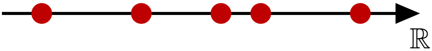

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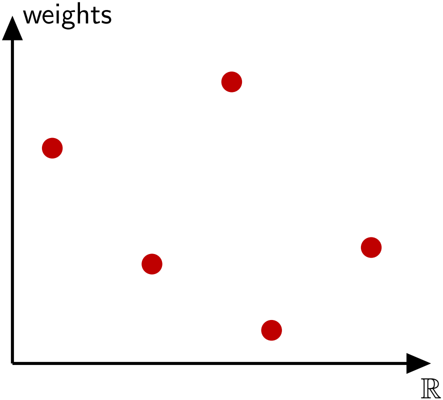

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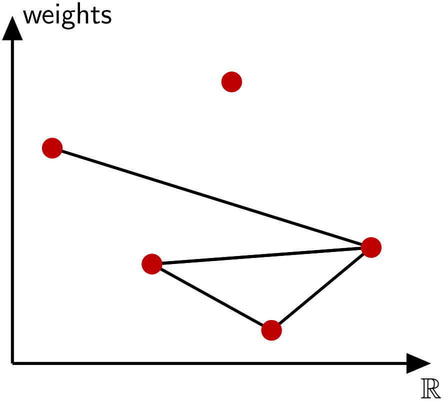

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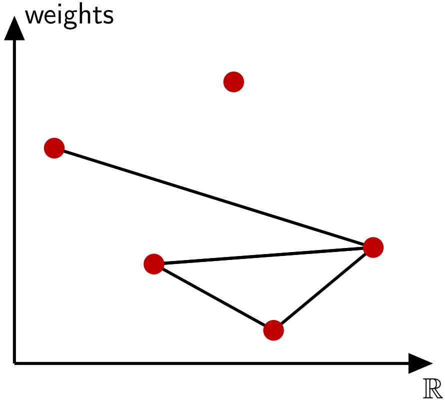

M. Deijfen, R. van der Hofstad, and G. Hooghiemstra. Scale-free percolation. *Ann. Inst. Henri Poincaré Probab. Stat.*, 49(3):817–838, 2013.

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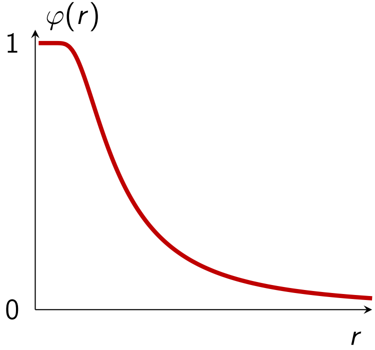
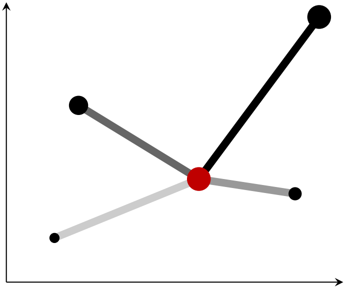
- regularity conditions on φ and F_W:

$$\lim_{r\uparrow\infty} \frac{\varphi(tr)}{\varphi(r)} = t^{-\alpha} \qquad \lim_{w\uparrow\infty} \frac{1-F_W(tw)}{1-F_W(w)} = t^{-\beta} \quad \alpha > 1,\ \beta \geqslant a\alpha$$

Example

$$\varphi(r) := 1 - \exp(-r^{-2}/\pi) \qquad \kappa(w_1, w_2) := w_1 w_2$$





Goal of the Study

- Spatial distribution of k-degree vertices:

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- Typical weight of a degree-k node:

$$\mathbb{P}(W \leqslant w_s) = 1/(2s)$$

Poisson Approximation

Theorem (Hirsch, Jahnel, Jhawar, J.; 2025)

ζ: *Poisson point process with intensity* $\mathbb{1}\{x \in [0,1]^d\}$

If

$$\begin{aligned} (sv_s)^{-1} w_s^{-\eta} &\in o(1/\log(s)) \\ F(w_s^\eta)\ w_s^{-(K+1)(1-\eta)} &\in o(1/\log(s)) \\ w_s^{-(1-K\delta)(1-\eta)} &\in o(1/\log(s)). \end{aligned}$$

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Then

$$d_{\mathrm{KR}}(\xi_s,\zeta) = \sup_{h\in \mathrm{LIP}(1)} \Big\{ \big| \mathbb{E}[h(\xi_s)] - \mathbb{E}[h(\zeta)] \big| \Big\} \xrightarrow[s\uparrow\infty]{} 0$$

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M. D. Penrose. Inhomogeneous random graphs, isolated vertices, and Poisson approximation. *J. Appl. Probab.*, 55(1):112–136, 2018.

S. K. Iyer and S. K. Jhawar. Poisson approximation and connectivity in a scale-free random connection model. *Electron. J. Probab.*, 26(23):Paper No. 86, 2021.

Stein's Method

O. Bobrowski, M. Schulte, and D. Yogeshwaran. Poisson process approximation under stabilization and Palm coupling. *Ann. H. Lebesgue*, 5:1489–1534, 2022.

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- Function
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Theorem (Bobrowski, Schulte, Yogeshwaran; 2022)

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Then

$$d_{\mathrm{KR}}(\xi, \zeta) \leqslant d_{\mathrm{TV}}(\lambda_\xi, \lambda_\zeta) + E_1 + E_2 + E_3 + E_4$$

Error Terms

Theorem (Bobrowski, Schulte, Yogeshwaran; 2022)

$$\hat{\mathcal{S}} : \mathbb{X}^m \to \mathcal{F} \qquad \boldsymbol{x} \subseteq \hat{\mathcal{S}}(\boldsymbol{x}) \qquad g_{\mathrm{tr}}(\boldsymbol{x}, \omega) := g(\boldsymbol{x}, \omega)\, \mathbb{1}\{\mathcal{S}(\boldsymbol{x}, \omega) \subseteq \hat{\mathcal{S}}(\boldsymbol{x})\}$$

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$$E_1 := \frac{2}{m!} \int_{\mathbb{X}^m} \mathbb{E}\big[g(\boldsymbol{x},\eta\cup\boldsymbol{x})\,\mathbb{1}\{\mathcal{S}(\boldsymbol{x},\eta\cup\boldsymbol{x}) \not\subset \hat{\mathcal{S}}(\boldsymbol{x})\}\big]\lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{x})$$

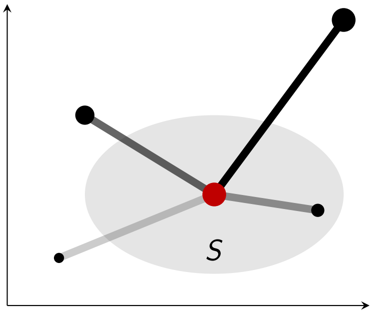
$$E_2 := \frac{2}{(m!)^2} \iint_{\mathbb{X}^m\times\mathbb{X}^m} \mathbb{1}\{\hat{\mathcal{S}}(\boldsymbol{x}) \cap \hat{\mathcal{S}}(\boldsymbol{z}) \neq \varnothing\} \times \mathbb{E}\big[g_{\mathrm{tr}}(\boldsymbol{x},\eta\cup\boldsymbol{x})\big]\,\mathbb{E}\big[g_{\mathrm{tr}}(\boldsymbol{z},\eta\cup\boldsymbol{z})\big]\lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{x})\lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{z})$$

$$E_3 := \frac{2}{(m!)^2} \iint_{\mathbb{X}^m\times\mathbb{X}^m} \mathbb{1}\{\hat{\mathcal{S}}(\boldsymbol{x}) \cap \hat{\mathcal{S}}(\boldsymbol{z}) \neq \varnothing\} \times \mathbb{E}\big[g_{\mathrm{tr}}(\boldsymbol{x},\eta\cup\boldsymbol{x}\cup\boldsymbol{z})g_{\mathrm{tr}}(\boldsymbol{z},\eta\cup\boldsymbol{x}\cup\boldsymbol{z})\big]\lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{x})\lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{z})$$

$$E_4 := \frac{2}{m!} \sum_{\varnothing\subsetneq I\subsetneq\{1,\dots,m\}} \frac{1}{(m-|I|)!} \int_{\mathbb{X}^m}\int_{\mathbb{X}^{m-|I|}} \mathbb{E}\big[g_{\mathrm{tr}}(\boldsymbol{x},\eta\cup\boldsymbol{x}\cup\boldsymbol{z}) \times g_{\mathrm{tr}}((\boldsymbol{x}_I,\boldsymbol{z}),\eta\cup\boldsymbol{x}\cup\boldsymbol{z})\big]\lambda_\xi^{\otimes m-|I|}(\mathrm{d}\boldsymbol{z})\lambda_\xi^{\otimes m}(\mathrm{d}\boldsymbol{x})$$

Proof Sketch

- Problem: g should be localized to $S \in \mathcal{F}$



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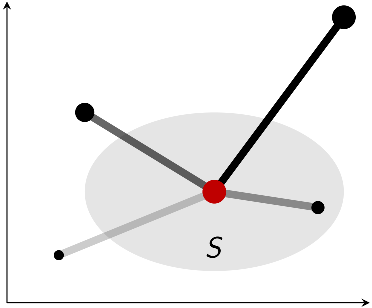

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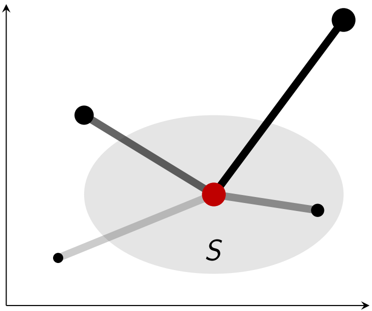
- Step 2: reach approximation: neglect far points



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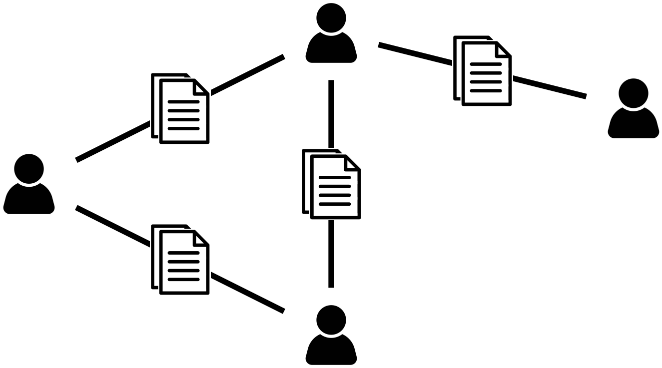

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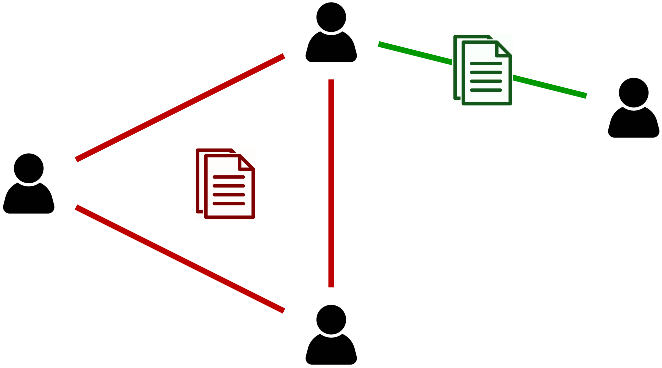

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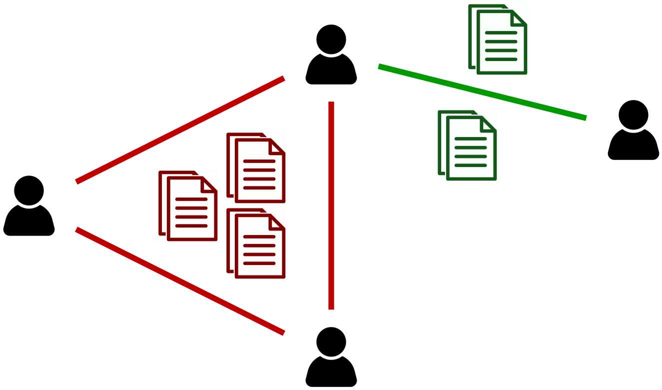

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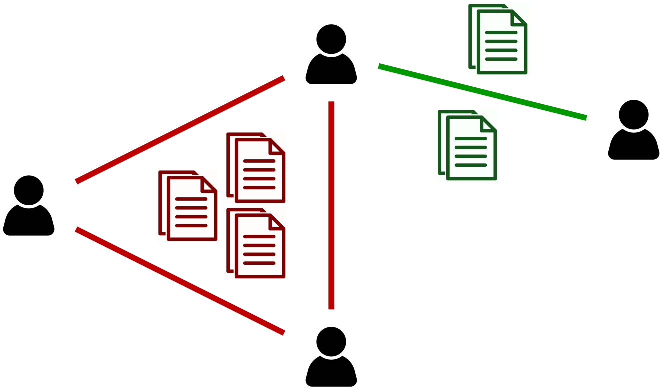


Simple graphs: information loss

Outline

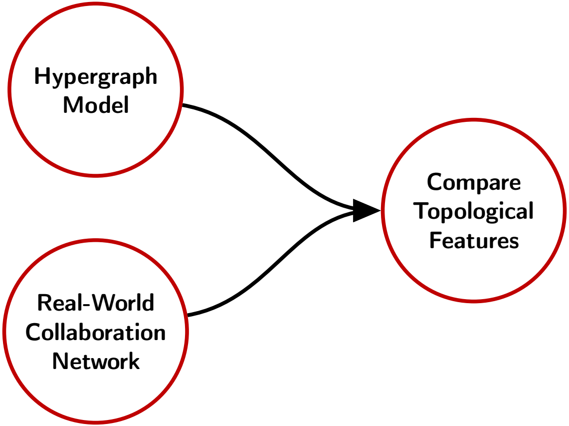
Hypergraph Model

Outline

Hypergraph Model

Real-World Collaboration Network

Outline


Hypergraph Model
Real-World Collaboration Network
Compare Topological Features

Random Connection Hypergraph Model (RCHM)

Two Poisson point processes
on $\mathbb{S} := \mathbb{R} \times [0, 1]$

- vertices: $\mathcal{P} \subset \mathbb{S}$
- hyperedges: $\mathcal{P}' \subset \mathbb{S}$

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Connections:

$$\{(X,U),(Y,V)\} \in \mathcal{P} \times \mathcal{P}'$$

$$|X - Y| \leqslant \beta\, U^{-\gamma}\, V^{-\gamma'} \qquad \gamma, \gamma' \in (0,1)$$

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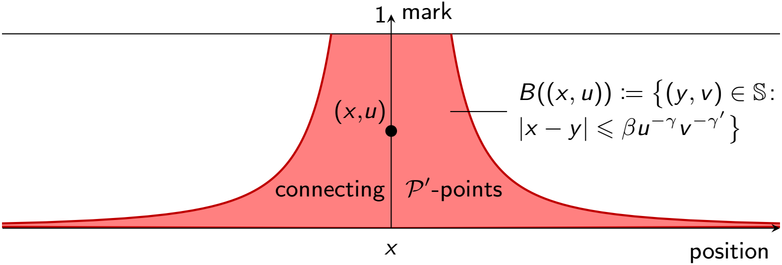
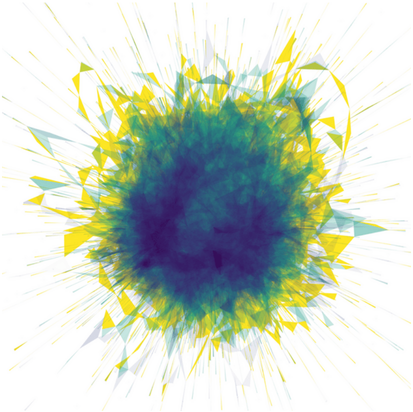


R. van der Hofstad, J. Komjáthy, and V. Vadon. Random intersection graphs with communities. *Adv. in Appl. Probab.*, 53(4):1061–1089, 2021.

Collaboration Network

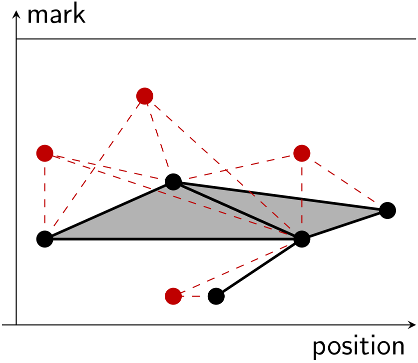
- Source: arXiv preprint server
- Vertices: $\sim 8 \cdot 10^5$ authors
- Hyperedges: $\sim 10^6$ documents



Statistics dataset

Higher-Order Degree Distribution

$$\deg(\Delta) := \mathcal{P}'\left(\bigcap_{p \in \Delta} B(p)\right)$$



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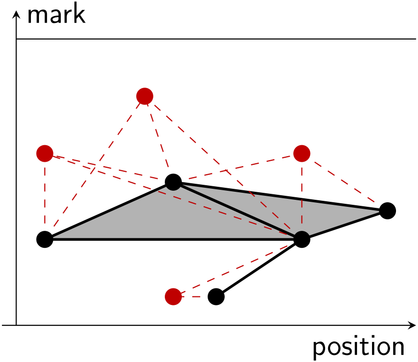

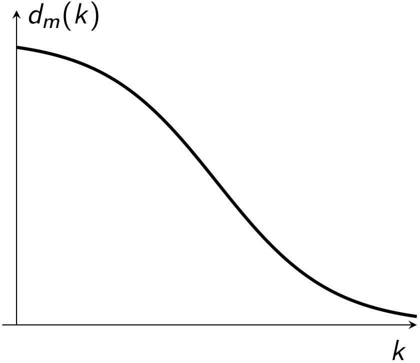

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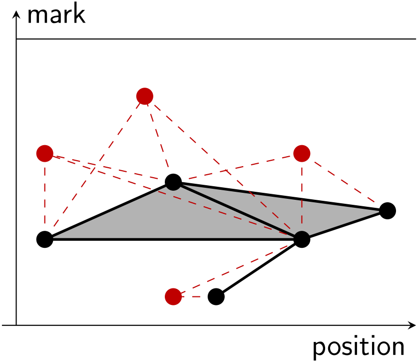

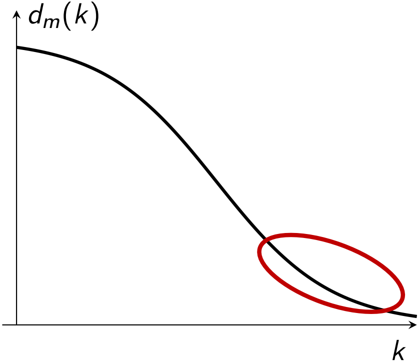

Higher-Order Degree Distributions

Theorem (Brun, Hirsch, J., Otto; 2025+)

$$\lim_{k\uparrow\infty} \log(d_m(k))/\log(k) = m - (m+1)/\gamma,$$

where

$$d_m(k) = \mathbb{P}\big(\deg(\Delta_m^*) \geqslant k\big).$$

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where

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- Parameter fitting: model parameters can be tuned to obtain a given power-law exponent.

Simplicial Complex

Idea: represent higher-order relationships with a simplicial complex

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Definition (Simplicial Complex)

- (P, Σ)
- $P \neq \varnothing$, $|P| < \infty$
- $\Sigma \subseteq 2^P$
- $p \in P \implies \{p\} \in \Sigma$
- $\sigma \in \Sigma$, $\Delta \subseteq \sigma \implies \Delta \in \Sigma$

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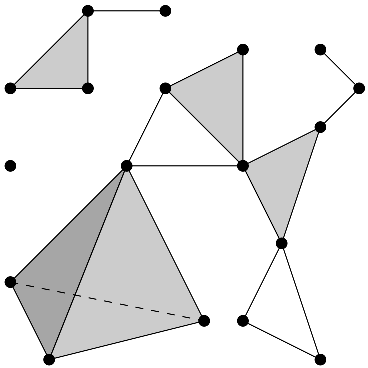

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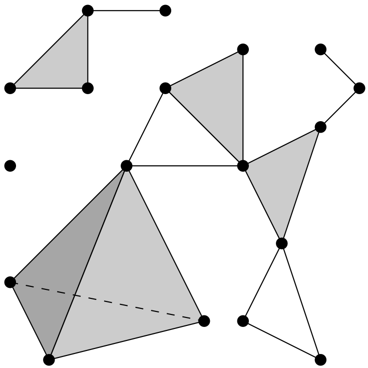
RCHM = simplicial complex

$$\Sigma = \left\{\Delta \subseteq \mathcal{P} : \deg(\Delta) \geqslant 1\right\}$$



Betti Numbers

Definition (k-chains)

$$C_k(K) := \left\{ \sum_i a_i \sigma_i, : a_i \in \mathbb{Z}_2,\ \sigma_i \in \Sigma, |\sigma_i| = k+1 \right\}$$

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$$\partial_k : C_k(K) \to C_{k-1}(K) \qquad \partial_k(\sigma_i) = \sum_{j=0}^{k} \sigma_i^{(j)}$$

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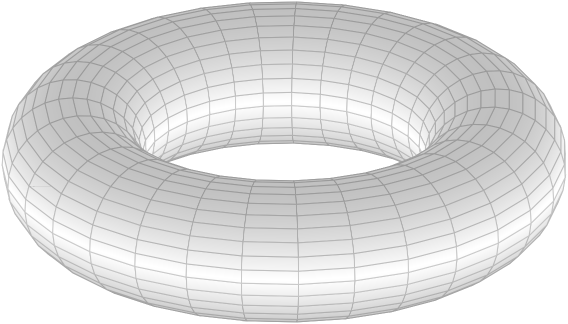
Definition (Cycle and boundary groups)

Cycle group: $Z_k = \ker(\partial_k)$

Boundary group: $B_k = \operatorname{im}(\partial_{k+1})$

kth Betti number: $\beta_k = \dim(Z_k) - \dim(B_k)$

Betti Numbers



$$\beta_0 = 1;\ \beta_1 = 2;\ \beta_2 = 1$$

CLT for Betti Numbers

- Window: $\mathbb{S}_n := [0, n] \times [0, 1]$
- Simplicial complex: $G_n := G(\mathcal{P} \cap \mathbb{S}_n, \mathcal{P}' \cap \mathbb{S}_n)$
- Betti number of dimension m in window n: $\beta_m^{(n)} := \beta_m(G_n)$

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Theorem (Brun, Hirsch, J., Otto; 2025+)

If $\gamma < 1/4$, $\gamma' < 1/(4(m+2))$, *then*

$$n^{-1/2}\left(\beta_m^{(n)} - \mathbb{E}[\beta_m^{(n)}]\right) \xrightarrow[n\uparrow\infty]{d} \mathcal{N}(0, \sigma^2) \qquad \sigma^2 \geqslant 0.$$

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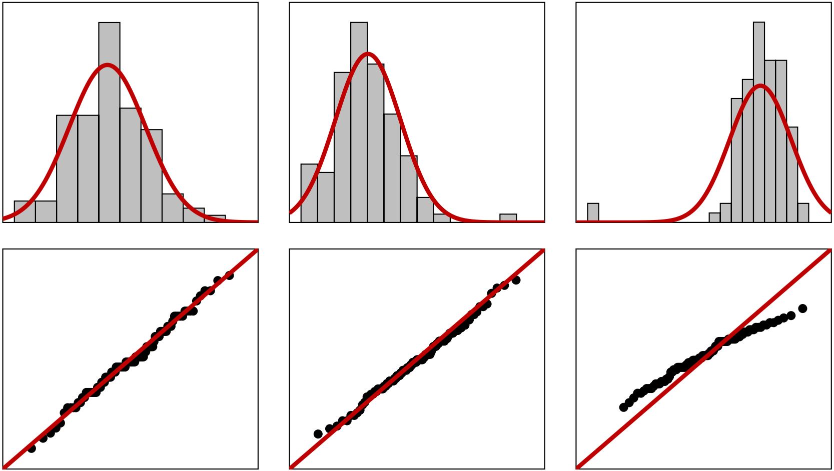
M. D. Penrose and J. E. Yukich. Central limit theorems for some graphs in computational geometry. *Ann. Appl. Probab.*, 11(4):1005–1041, 2001.

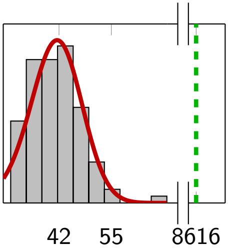
D. Pabst. Betti numbers in the random connection model for higher-dimensional simplicial complexes and the Boolean model. *arXiv preprint arXiv:2506.13429*, 2025.

Simulated Distribution of Betti-1 Numbers



$\gamma = 0.25$ $\gamma = 0.50$ $\gamma = 0.75$

Hypothesis Test



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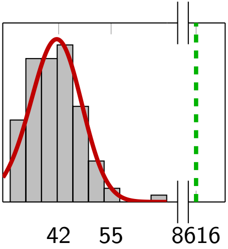

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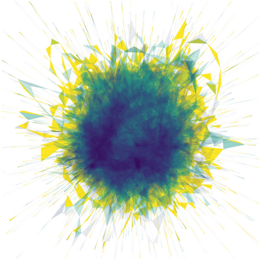
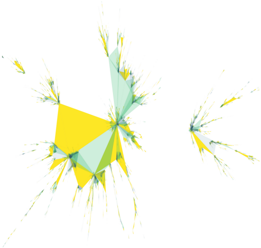


Network data



Simulation

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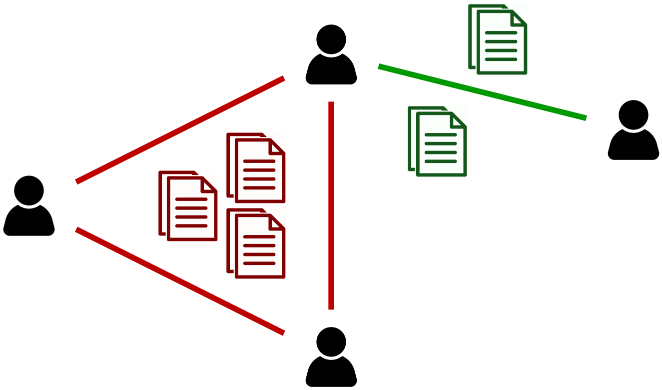
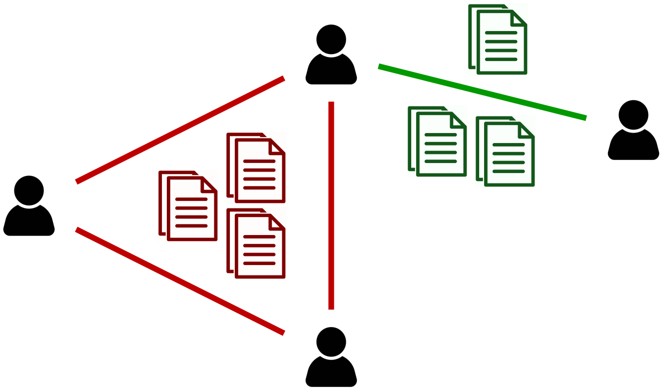
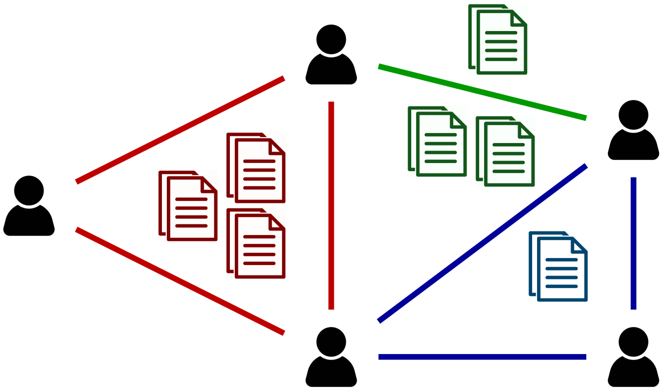
Complex Systems



Complex Systems



Complex Systems



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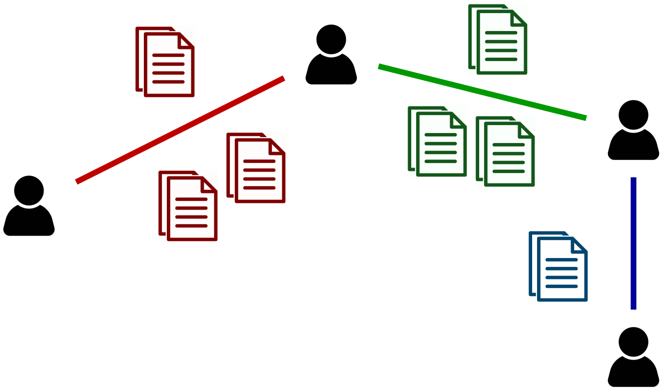

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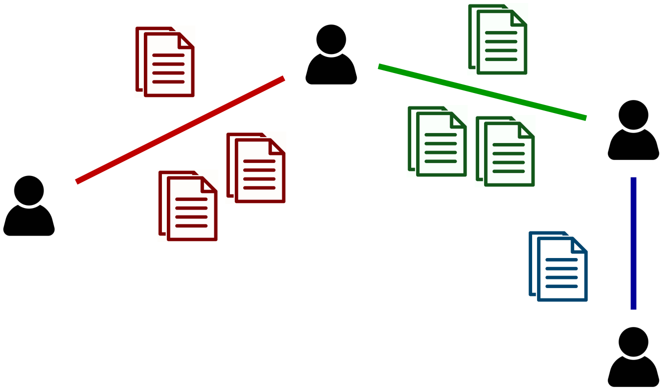

Network dynamics?

Dynamic Random Connection Hypergraph Model

Vertices: $\mathcal{P}$

position	$X \in \mathbb{R}$	Leb
mark	$U \in (0, 1]$	Leb
birth time	$B \in \mathbb{R}$	Leb
lifetime	$L \in [0, \infty)$	Exp(1)

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Connection conditions:

spatial condition

$$|X - Z| \leqslant \beta U^{-\gamma} W^{-\gamma'}$$

temporal condition

$$B \leqslant R \leqslant B + L$$

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E. Onaran, O. Bobrowski, and R. J. Adler. Functional central limit theorems for local statistics of spatial birth–death processes in the thermodynamic regime. *Ann. Appl. Probab.*, 33(5):3958–3986, 2023.

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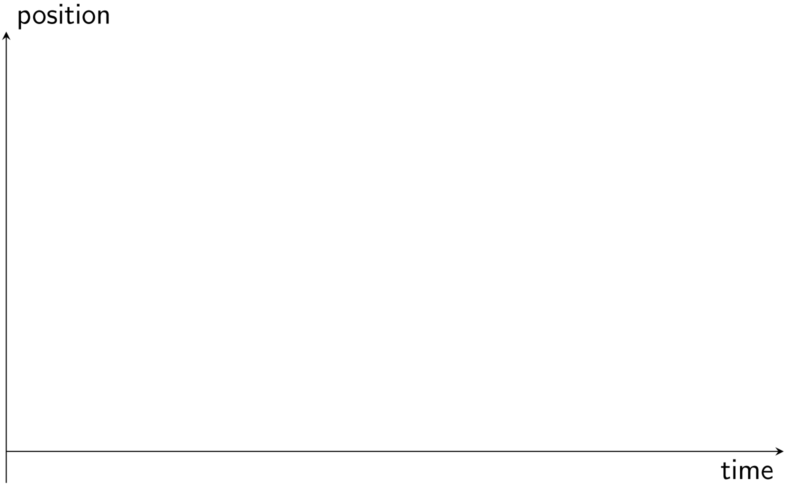

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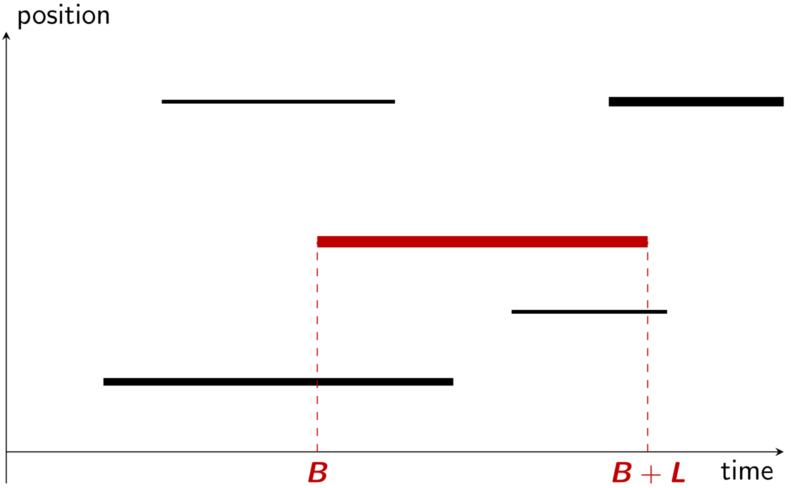

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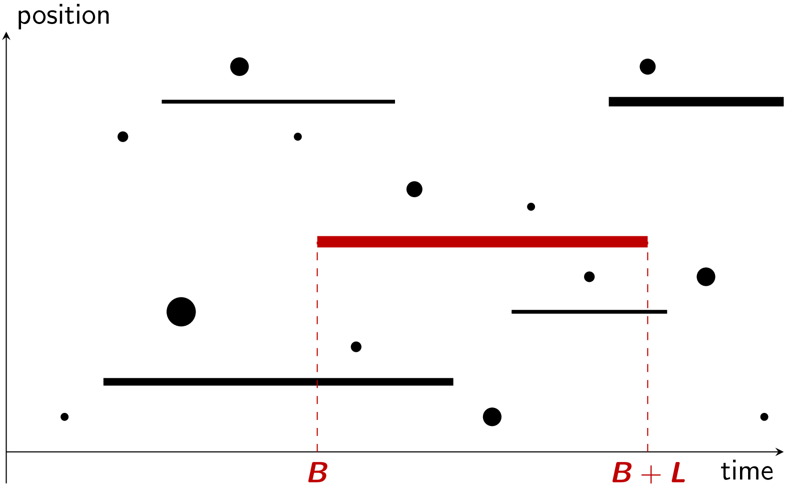

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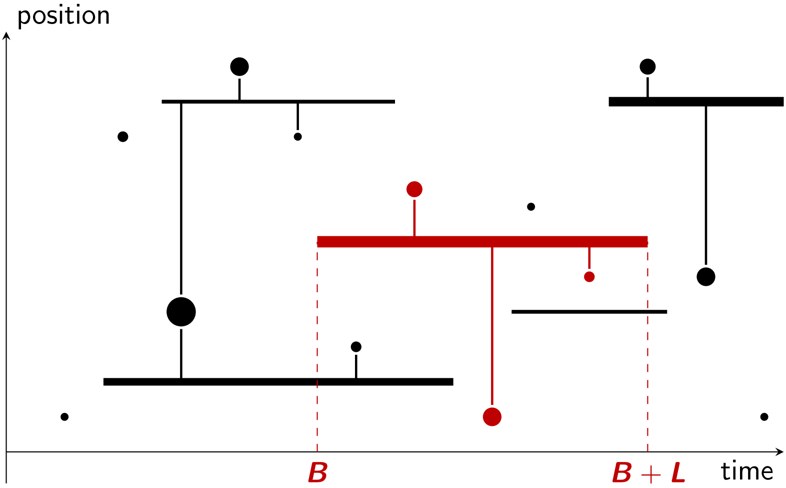

Edge counts

- Degree of $P := (X, U, B, L) \in \mathcal{P}$:

$$\deg(P;t) := \sum_{(Z,W,R)\in\mathcal{P}'} \mathbb{1}\{|X-Z| \leqslant \beta U^{-\gamma} W^{-\gamma'}\} \times \mathbb{1}\{B \leqslant R \leqslant t \leqslant B+L\}$$

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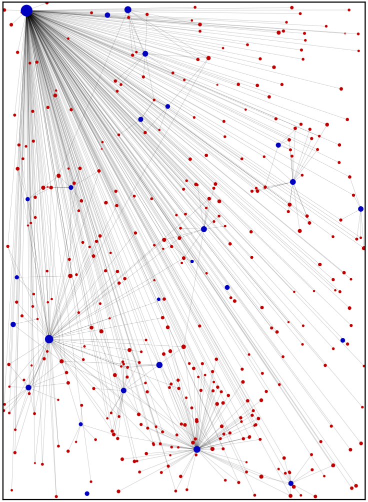
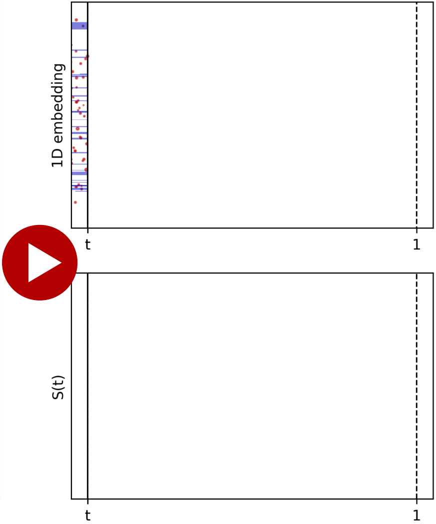
- Edge count

$$S_n(\cdot) := \sum_{P\in\mathcal{P}} \deg(P; \cdot)\, \mathbb{1}\{X \in [0, n]\}$$

Edge counts





Finite Variance Case

Proposition (Hirsch, Jahnel, J.; 2025+)

If $\gamma < 1/2$ *and* $\gamma' \in (0,1)$*, then*

$$\overline{S}_n(t) := n^{-1/2}\big(S_n(t) - \mathbb{E}[S_n(t)]\big) \xrightarrow[n\uparrow\infty]{d} \mathcal{N}(0,\sigma^2) \quad \sigma^2 > 0.$$

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Theorem (Hirsch, Jahnel, J.; 2025+)

G: Gaussian process

$$\mathrm{Cov}(G(t_1), G(t_2)) = (c_1 + c_2|t_1 - t_2|)\exp(-\mu|t_1 - t_2|)$$

If $\gamma, \gamma' < 1/4$, then

$$\overline{S}_n(\cdot) \xrightarrow[n\uparrow\infty]{d} G(\cdot).$$

Univariate Stable Limit Theorem

Theorem (Hirsch, Jahnel, J.; 2025+)

If $\gamma > 1/2$, *then*

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Functional Convergence of Edge Counts

- $\nu([\varepsilon, \infty)) := c\varepsilon^{-1/\gamma}$: Lévy measure on $[0, \infty)$
- $\mathcal{P}_\infty := \mathrm{PPP}(\nu \otimes \mathrm{Leb}_{\mathbb{R}} \otimes \mathrm{Exp}(1))$: Poisson point process on $[0, \infty) \times \mathbb{R} \times \mathbb{R}_+$

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$$S^*_\varepsilon(\,\cdot\,) := \sum_{(J,B,L) \in \mathcal{P}_\infty} J(\cdot - B)\, \mathbb{1}\{J \geqslant \tilde{c}\varepsilon^\gamma\}\, \mathbb{1}\{B \leqslant \cdot \leqslant B + L\}$$

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Theorem (Hirsch, Jahnel, J.; 2025+)

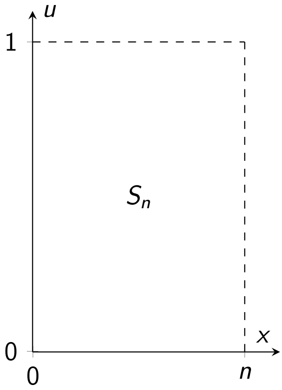
If $\gamma > 1/2$ and $\gamma' < 1/4$, then $\overline{S}_\infty(\cdot) \in D([0,1],\mathbb{R})$, and

$$\overline{S}_n(\cdot) := n^{-\gamma}\big(S_n(t) - \mathbb{E}[S_n(t)]\big) \xrightarrow[n\uparrow\infty]{d} \overline{S}_\infty(\cdot) \quad \textit{in } D([0,1],\mathbb{R}).$$

Proof of Functional SLT

Step 1

$$S_n(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \deg(P;\,\cdot\,)$$



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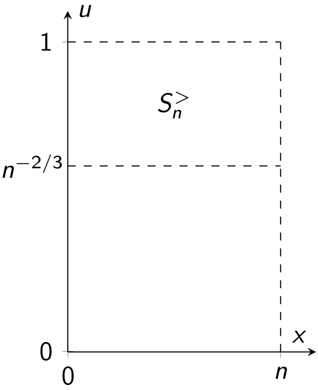

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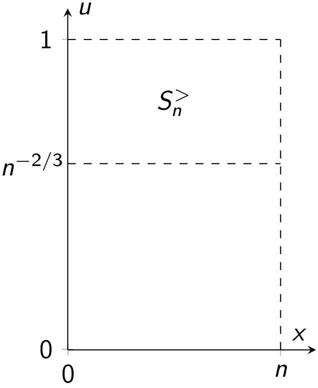
Proposition (Hirsch, Jahnel, J.; 2025+)

If $\gamma > 1/2$, $\gamma' < 1/4$, *then*

$$\overline{S}_n^>(\,\cdot\,) \xrightarrow[n\uparrow\infty]{d} 0 \qquad \textit{in } D([0,1],\mathbb{R}).$$



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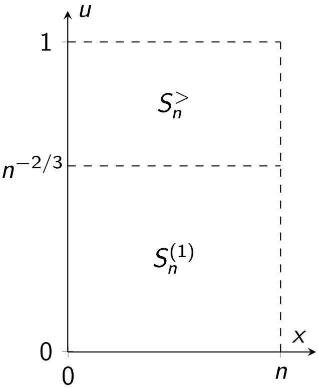

Step 2

$$S_n^{(2)}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \mathbb{E}\big[\deg(P;\,\cdot\,) \bigm| P\big]\, \mathbb{1}\{U \leqslant n^{-2/3}\}$$

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Proposition (Hirsch, Jahnel, J.; 2025+)

If $\gamma > 1/2$, *then* $\big\|\overline{S}_n^{(1)} - \overline{S}_n^{(2)}\big\| \xrightarrow[n\uparrow\infty]{\mathbb{P}} 0.$

Part 2 – Steps

$$\overline{S}_n^{(2)} \xrightarrow{\text{goal}} \overline{S}_\infty$$

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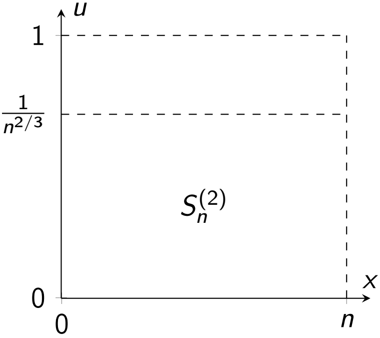

Part 2 – Steps

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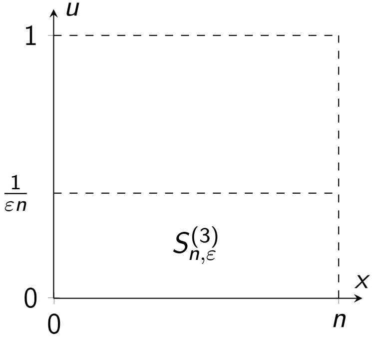

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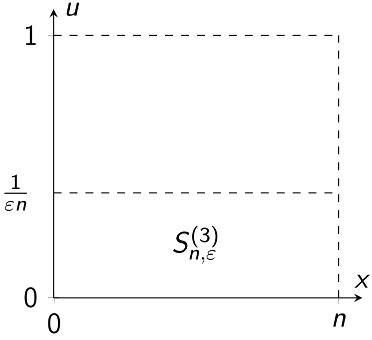
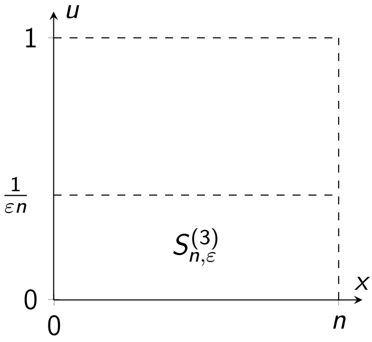


$$S_{n,\varepsilon}^{(3)}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \mathbb{E}[\deg(P;\,\cdot\,) \mid P]\, \mathbb{1}\{U \leqslant 1/(\varepsilon n)\}$$

Part 2 – Steps

$$\begin{tikzcd}
\overline{S}^{(3)}_{n,\varepsilon} \arrow[d, "\text{Step 3}", "\varepsilon \to 0\ (\forall n)"'] & \\
\overline{S}^{(2)}_{n} \arrow[r, dashed, "\text{goal}"] & \overline{S}_{\infty}
\end{tikzcd}$$



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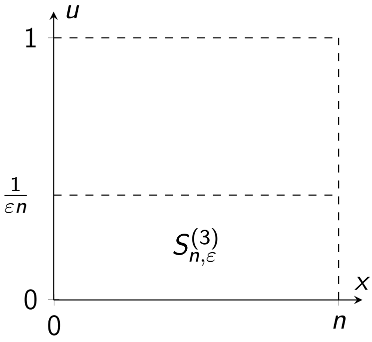
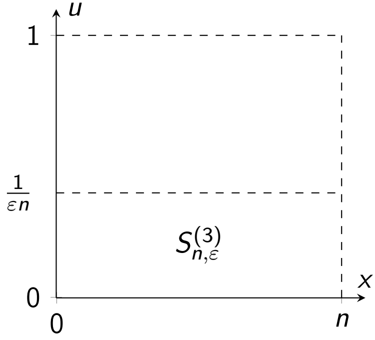
$$\begin{tikzcd}
\overline{S}^{(3)}_{n,\varepsilon} \arrow[rr, "n \to \infty", "\text{Step 4}"'] \arrow[d, "\varepsilon \to 0\ (\forall n)"', "\text{Step 3}"] & & \overline{S}^{*}_{\varepsilon} \\
\overline{S}^{(2)}_{n} \arrow[rr, dashed, "\text{goal}"] & & \overline{S}_{\infty}
\end{tikzcd}$$



$$S^{(3)}_{n,\varepsilon}(\,\cdot\,) := \sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \mathbb{E}[\deg(P;\,\cdot\,) \mid P]\, \mathbb{1}\{U \leqslant 1/(\varepsilon n)\}$$

Part 2 – Steps

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overline{S}^{(3)}_{n,\varepsilon} \arrow[r, "n \to \infty", "\text{Step 4}"'] \arrow[d, "\varepsilon \to 0\ (\forall n)"', "\text{Step 3}"] & \overline{S}^{*}_{\varepsilon} \arrow[d, "\text{Step 5}"', "\varepsilon \to 0"] \\
\overline{S}^{(2)}_{n} \arrow[r, dashed, "\text{goal}"] & \overline{S}_{\infty}
\end{tikzcd}$$



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Step 3

Goal:

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\uparrow\infty}\mathbb{P}\Big(d_{\mathrm{Sk}}\big(\overline{S}^{(3)}_{n,\varepsilon},\overline{S}^{(2)}_{n}\big)>\delta\Big)=0\qquad\text{for all } \delta>0$$

Proposition (Hirsch, Jahnel, J.; 2025+)

Let $\gamma>1/2$ *and* $\gamma'\in(0,1)$. *Then,*

$$\limsup_{n\uparrow\infty}\Big\|\overline{S}^{(3)}_{n,\varepsilon}-\overline{S}^{(2)}_{n}\Big\|\xrightarrow[\varepsilon\downarrow 0]{\mathbb{P}}0.$$

Step 4

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\overline{S}^{(3)}_{n,\varepsilon} \arrow[r, "n \to \infty", "\text{Step 4}"'] \arrow[d, "\varepsilon \to 0 \quad (\forall n) \text{ unif.}"', "\text{Step 3}"] & \overline{S}^{*}_{\varepsilon} \arrow[d, "\text{Step 5}"', "\varepsilon \to 0"] \\
\overline{S}^{(2)}_{n} \arrow[r, dashed, "\text{goal}"] & \overline{S}_{\infty}
\end{tikzcd}
$$

Step 4

Notation: $N_s(p) := \{(z, w) \in \mathbb{R} \times (0, 1] \colon |x - z| \leqslant \beta u^{-\gamma} w^{-\gamma'}\}$

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In $S_n^{(3)}$: $\quad U \leqslant 1/(\varepsilon n) \quad \Longleftrightarrow \quad n^{-\gamma}|N_s(P_s)| \geqslant \tilde{c}\varepsilon^{\gamma}$

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In $S_n^{(3)}$: $U \leqslant 1/(\varepsilon n) \iff n^{-\gamma}|N_s(P_s)| \geqslant \tilde{c}\varepsilon^{\gamma}$

Lemma (Hirsch, Jahnel, J.; 2025+)

Let $\gamma > 1/2$ and $\gamma' \in (0,1)$. Then,

$$n\,\mathbb{P}\big(n^{-\gamma}|N_s(P)| \in \cdot\,\big) \xrightarrow[n\uparrow\infty]{v} \nu(\,\cdot\,) \qquad \nu([a,\infty)) := \tilde{c}^{1/\gamma} a^{-1/\gamma}.$$

S. I. Resnick. *Heavy-Tail Phenomena: Probabilistic and Statistical Modeling.* Springer, New York, 2007.

Step 4 – Convergence of Point Processes

$$\sum_{P \in \mathcal{P} \cap \mathbb{S}_n} \delta_{(n^{-\gamma}|N_{\mathbf{s}}(P_{\mathbf{s}})|, B, L)} \xrightarrow[n \to \infty]{\mathrm{v}} \mathcal{P}_\infty = \mathrm{PPP}(\nu \otimes \mathrm{Leb}_{\mathbb{R}} \otimes \mathrm{Exp}(1))$$

Step 4 – Convergence of Point Processes

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\begin{array}{ccc}
\displaystyle\sum_{P\in\mathcal{P}\cap\mathbb{S}_n} \delta_{(n^{-\gamma}|N_{\mathbf{s}}(P_{\mathbf{s}})|,B,L)} & \xrightarrow[n\to\infty]{\mathrm{v}} & \mathcal{P}_\infty = \mathrm{PPP}(\nu\otimes \mathrm{Leb}_{\mathbb{R}}\otimes \mathrm{Exp}(1)) \\
\Big\downarrow \chi & & \\
\end{array}
$$

$$
\begin{aligned}
\overline{S}^{(3)}_{n,\varepsilon}(\,\cdot\,) = \sum_{P\in\mathcal{P}\cap\mathbb{S}_n} & n^{-\gamma}|N_{\mathrm{s}}(P_{\mathrm{s}})|(\cdot\, - B) \\
& \times \mathbb{1}\{n^{-\gamma}|N_{\mathrm{s}}(P_{\mathrm{s}})| \geqslant \tilde{c}\varepsilon^{\gamma}\} \\
& \times \mathbb{1}\{B \leqslant \,\cdot\, \leqslant B+L\} \\
-n^{-\gamma}\, & \mathbb{E}[S^{(3)}_{n,\varepsilon}(\,\cdot\,)]
\end{aligned}
$$

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$$\begin{aligned}\overline{S}^{(3)}_{n,\varepsilon}(\cdot) = \sum_{P \in \mathcal{P} \cap \mathbb{S}_n} & n^{-\gamma}|N_{\mathrm{s}}(P_{\mathrm{s}})|(\cdot - B) \\ & \times \mathbb{1}\{n^{-\gamma}|N_{\mathrm{s}}(P_{\mathrm{s}})| \geqslant \tilde{c}\varepsilon^{\gamma}\} \\ & \times \mathbb{1}\{B \leqslant \cdot \leqslant B + L\} \\ & - n^{-\gamma}\,\mathbb{E}[S^{(3)}_{n,\varepsilon}(\cdot)]\end{aligned}$$

$$\begin{aligned}\overline{S}^{*}_{\varepsilon}(\cdot) = \sum_{(J,B,L) \in \mathcal{P}_\infty} & J(\cdot - B) \\ & \times \mathbb{1}\{J \geqslant \tilde{c}\varepsilon^{\gamma}\} \\ & \times \mathbb{1}\{B \leqslant \cdot \leqslant B + L\} \\ & - \mathbb{E}[S^{*}_{\varepsilon}(\cdot)]\end{aligned}$$

Step 4 – Convergence of Point Processes

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\begin{array}{ccc}
\displaystyle\sum_{P\in\mathcal{P}\cap\mathbb{S}_n} \delta_{(n^{-\gamma}|N_{\mathbf{s}}(P_{\mathbf{s}})|,B,L)} & \xrightarrow[n\to\infty]{\mathrm{v}} & \mathcal{P}_\infty = \mathrm{PPP}(\nu\otimes\mathrm{Leb}_{\mathbb{R}}\otimes\mathrm{Exp}(1)) \\
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& \times \mathbb{1}\{B \leqslant \,\cdot\, \leqslant B+L\} \\
& - n^{-\gamma}\,\mathbb{E}[S^{(3)}_{n,\varepsilon}(\,\cdot\,)]
\end{aligned}
& &
\begin{aligned}
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& - \mathbb{E}[S^{*}_{\varepsilon}(\,\cdot\,)]
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\end{array}
$$

Proposition (Hirsch, Jahnel, J.; 2025+)

χ is almost surely continuous with respect to the distribution of $\mathcal{P}_\infty$.

Step 4 – Convergence of Point Processes

$$\begin{tikzcd}
\displaystyle\sum_{P\in\mathcal{P}\cap\mathbb{S}_n} \delta_{(n^{-\gamma}|N_{\mathbf{s}}(P_{\mathbf{s}})|,B,L)} \arrow[r, "\mathrm{v}", "n\to\infty"'] \arrow[d, "\chi"'] & \mathcal{P}_\infty = \mathrm{PPP}(\nu\otimes \mathrm{Leb}_{\mathbb{R}}\otimes \mathrm{Exp}(1)) \arrow[d, "\chi"'] \\
\begin{aligned}\overline{S}^{(3)}_{n,\varepsilon}(\,\cdot\,) = \sum_{P\in\mathcal{P}\cap\mathbb{S}_n} & n^{-\gamma}|N_{\mathrm{s}}(P_{\mathrm{s}})|(\,\cdot\, - B)\\ &\times \mathbb{1}\{n^{-\gamma}|N_{\mathrm{s}}(P_{\mathrm{s}})|\geqslant \tilde{c}\varepsilon^\gamma\}\\ &\times \mathbb{1}\{B\leqslant \,\cdot\, \leqslant B+L\}\\ & -n^{-\gamma}\,\mathbb{E}[S^{(3)}_{n,\varepsilon}(\,\cdot\,)]\end{aligned} \arrow[r, bend left, "\mathrm{d}", "n\uparrow\infty"'] & \begin{aligned}\overline{S}^{*}_{\varepsilon}(\,\cdot\,) = \sum_{(J,B,L)\in\mathcal{P}_\infty} & J(\,\cdot\, - B)\\ &\times \mathbb{1}\{J\geqslant \tilde{c}\varepsilon^\gamma\}\\ &\times \mathbb{1}\{B\leqslant \,\cdot\, \leqslant B+L\}\\ & -\mathbb{E}[S^{*}_{\varepsilon}(\,\cdot\,)]\end{aligned}
\end{tikzcd}$$

Proposition (Hirsch, Jahnel, J.; 2025+)

χ *is almost surely continuous with respect to the distribution of* $\mathcal{P}_\infty$.

Step 5

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overline{S}^{(3)}_{n,\varepsilon} \arrow[r, "n \to \infty", "\text{Step 4}"'] \arrow[d, "\varepsilon \to 0 \quad (\forall n) \text{ unif.}"', "\text{Step 3}"] & \overline{S}^{*}_{\varepsilon} \arrow[d, "\text{Step 5}"', "\varepsilon \to 0"] \\
\overline{S}^{(2)}_{n} \arrow[r, dashed, "\text{goal}"] & \overline{S}_{\infty}
\end{tikzcd}$$

Step 5 – Almost Sure Convergence of $\overline{S}^*_\varepsilon$

Proposition (Hirsch, Jahnel, J.; 2025+)

Let $\varepsilon_k \to 0$ be a decreasing sequence. Then,

$$\sup_{n,m \geqslant N} \left\| \overline{S}^*_{\varepsilon_n} - \overline{S}^*_{\varepsilon_m} \right\| \xrightarrow[N \uparrow \infty]{\text{a.s.}} 0.$$

Step 5

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\overline{S}^{(3)}_{n,\varepsilon} \arrow[r, "n \to \infty", "\text{Step 4}"'] \arrow[d, "\varepsilon \to 0 \quad (\forall n)\ \text{unif.}"', "\text{Step 3}"] & \overline{S}^{*}_{\varepsilon} \arrow[d, "\text{Step 5}"', "\varepsilon \to 0"] \\
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Summary

- **Weighted Random Connection Models**
  - simple network model
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  - functional CLT
  - functional SLT

Thank you for your attention!

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References

M. Brun, C. Hirsch, P. Juhász, and M. Otto.
Random connection hypergraphs.
arXiv preprint arXiv:2407.16334v2, 2025.

C. Hirsch, B. Jahnel, S. K. Jhawar, and P. Juhász.
Poisson approximation of fixed-degree nodes in weighted random connection models.
Stochastic Process. Appl., 183:104593, 2025.

C. Hirsch, B. Jahnel, and P. Juhász.
Functional limit theorems for edge counts in dynamic random connection hypergraphs.
arXiv preprint arXiv:2507.16270, 2025.

C. Hirsch and P. Juhász.
Network simulator.
https://github.com/shepherd92/network_simulator, 2025.

C. Hirsch and P. Juhász.
On the topology of higher-order age-dependent random connection models.
Methodol. Comput. Appl. Probab., 27(2):Paper No. 44, 41, 2025.