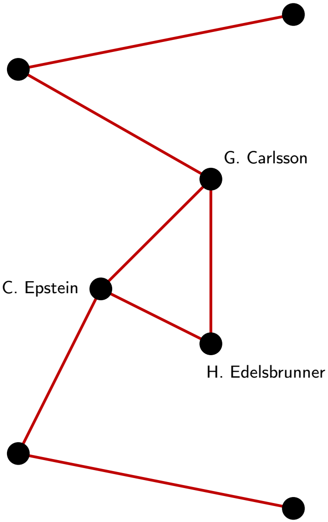
On the Topology of Higher-Order Age-Dependent Random Connection Models

Christian Hirsch, Peter Juhasz

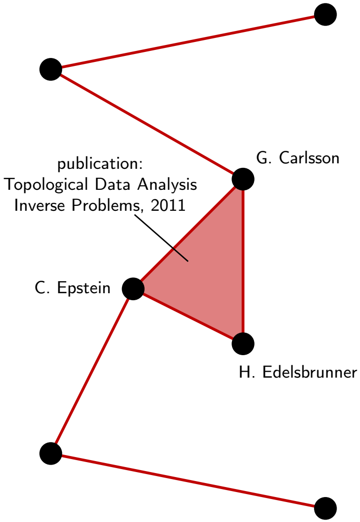
Department of Mathematics
Aarhus University

September 8, 2023

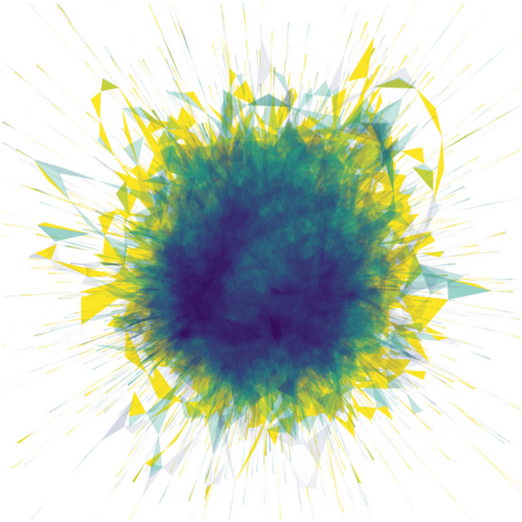
Higher-Order Networks



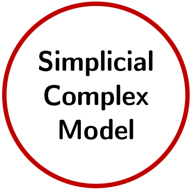
Higher-Order Networks



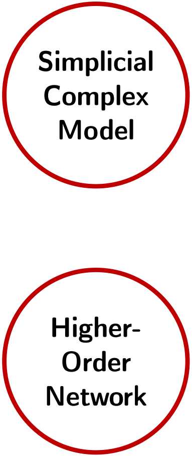
Publications of Authors in Statistics



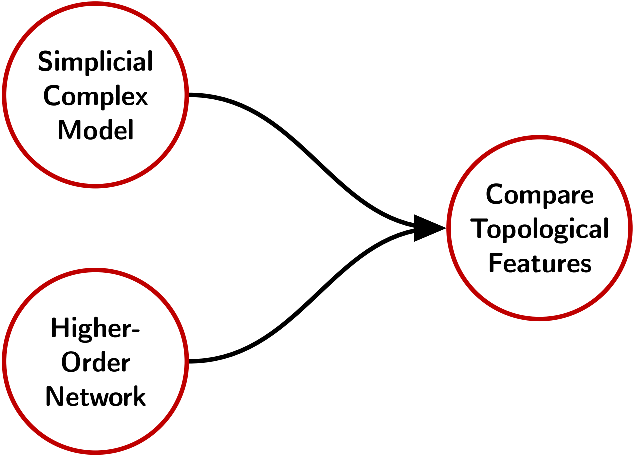
Goal



Goal



Goal


Simplicial
Complex
Model
Higher-
Order
Network
Compare
Topological
Features

Age-Dependent Random Connection Model

Gracar, Peter, et al. "The age-dependent random connection model." *Queueing Systems* 93 (2019): 309-331.

Age-Dependent Random Connection Model

Gracar, Peter, et al. "The age-dependent random connection model." *Queueing Systems* 93 (2019): 309-331.

- Vertices: Poisson process $\mathcal{P}\{(x_i,\ t_i)\} \subset \mathbb{R} \times [0,1]$

Age-Dependent Random Connection Model

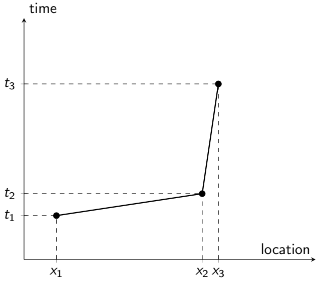
Gracar, Peter, et al. "The age-dependent random connection model." *Queueing Systems* 93 (2019): 309-331.

- Vertices: Poisson process $\mathcal{P}\{(x_i,\ t_i)\} \subset \mathbb{R} \times [0,1]$
- Connections: $\delta_{ij} \leq \frac{1}{2}\, t_i^{-\gamma}\, t_j^{\gamma-1}$

Age-Dependent Random Connection Model

Gracar, Peter, et al. "The age-dependent random connection model." *Queueing Systems* 93 (2019): 309-331.

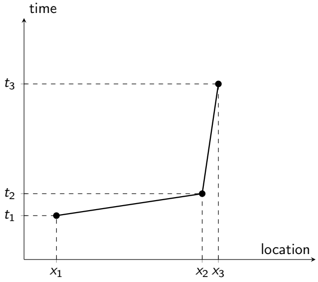
- Vertices: Poisson process $\mathcal{P}\{(x_i,\ t_i)\} \subset \mathbb{R} \times [0,1]$
- Connections: $\delta_{ij} \leq \frac{1}{2}\, t_i^{-\gamma}\, t_j^{\gamma-1}$



Age-Dependent Random Connection Model

Gracar, Peter, et al. "The age-dependent random connection model." *Queueing Systems* 93 (2019): 309-331.

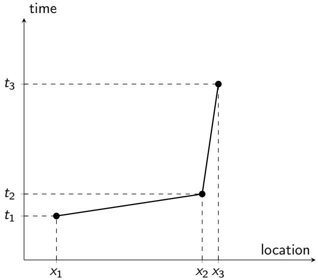
- Vertices: Poisson process $\mathcal{P}\{(x_i,\ t_i)\} \subset \mathbb{R} \times [0,1]$
- Connections: $\delta_{ij} \leq \frac{1}{2}\, t_i^{-\gamma}\, t_j^{\gamma-1}$
- Click for Graph



Age-Dependent Random Connection Model

Gracar, Peter, et al. "The age-dependent random connection model." *Queueing Systems* 93 (2019): 309-331.

- Vertices: Poisson process $\mathcal{P}\{(x_i,\ t_i)\} \subset \mathbb{R} \times [0,1]$
- Connections: $\delta_{ij} \leq \frac{1}{2}\, t_i^{-\gamma}\ t_j^{\gamma-1}$
- Click for Graph



Properties if $\gamma \in \left(\frac{1}{2}, 1\right)$:

- ✓ sparse
- ✓ scale-free
- ✓ ultra small
- ✓ high clustering
- × graphical model

Research Outline

- Consider ADRCM as a “clique complex”

Research Outline

- Consider ADRCM as a “clique complex”
- Determine characteristics of ADRCM as a higher-order network

Research Outline

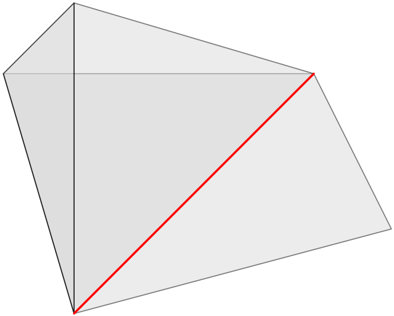
- Consider ADRCM as a “clique complex”
- Determine characteristics of ADRCM as a higher-order network
- Extend ADRCM to match a larger number of characteristics

Research Outline

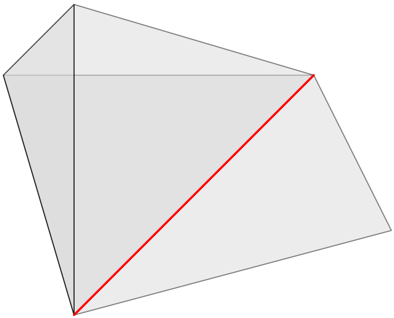
- Consider ADRCM as a "clique complex"
- Determine characteristics of ADRCM as a higher-order network
- Extend ADRCM to match a larger number of characteristics
- Illustrate the findings on simulated networks and on real data

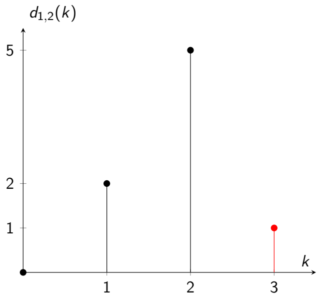
Higher-Order Degree Distributions

Higher-Order Degree Distributions



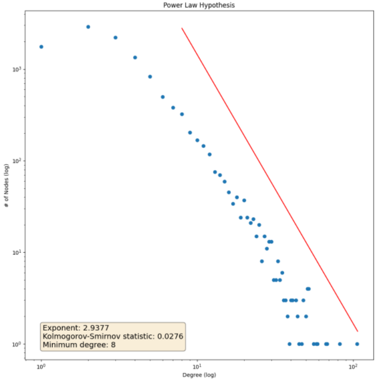
Higher-Order Degree Distributions

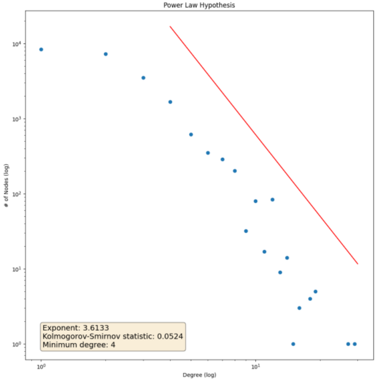




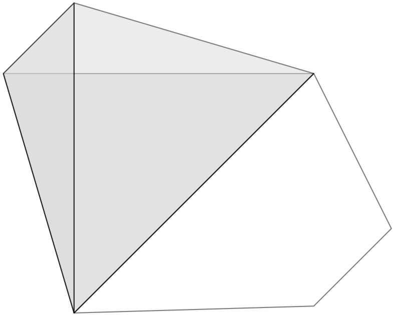
$$\lim_{k \to \infty} d_{m,m'}(k) \sim k^{m-1-\frac{m+1}{\gamma}}$$

Higher-Order Degree Distribution – Statistics Theory

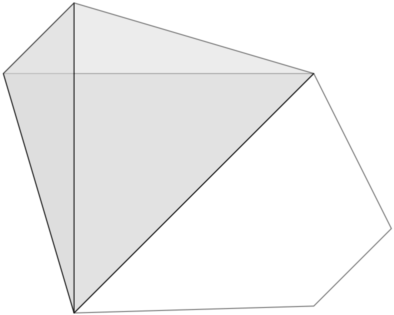




CLT for Betti Numbers

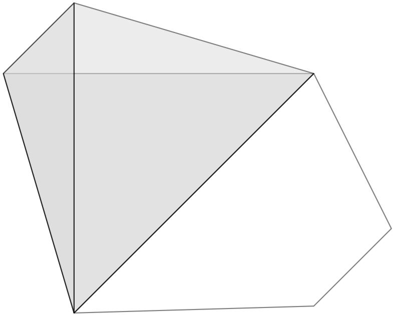


CLT for Betti Numbers



- $\gamma \ll 1$

CLT for Betti Numbers



- $\gamma \ll 1$
- $\frac{\beta_n - \mathbb{E}[\beta_n]}{\sqrt{\mathrm{Var}(\beta_n)}} \xrightarrow{d} \mathcal{N}(0,1)$

Distribution of Edge Count

Distribution of Edge Count

CLT for Edge Count

- $\gamma < 1/2$
- S_n : number of m-simplices in the interval $[0, n]$
- $\frac{S_n - \mathbb{E}[S_n]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$

Distribution of Edge Count

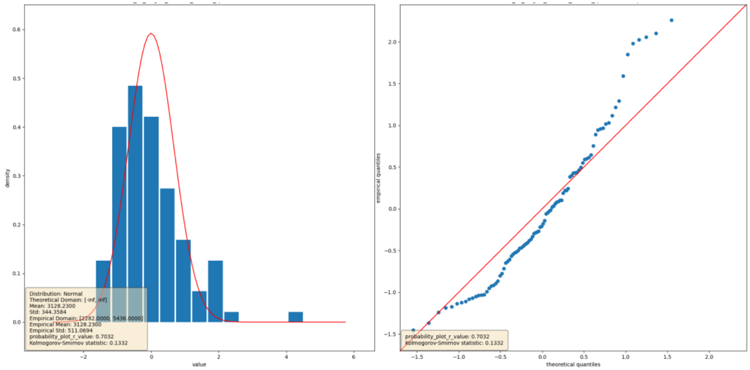
CLT for Edge Count

- $\gamma < 1/2$
- S_n : number of m-simplices in the interval $[0, n]$
- $\frac{S_n - \mathbb{E}[S_n]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$

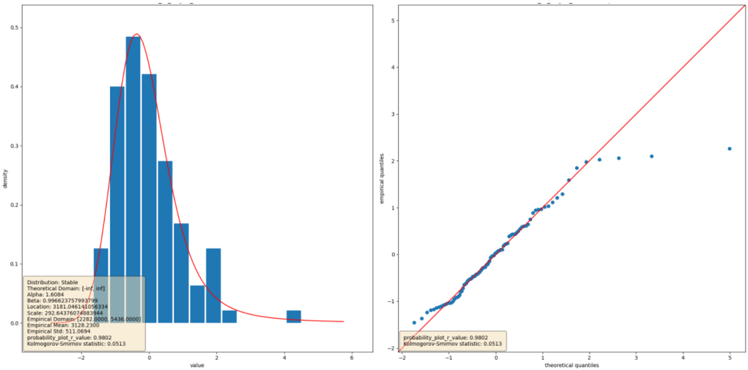
Stable Limit Law for Edge Count

- $1/2 < \gamma < 1$
- S_n : number of edges in the interval $[0, n]$
- $\frac{S_n - \mathbb{E}[S_n]}{n^\gamma} \xrightarrow{d} \mathcal{S}(1/\gamma)$

Stable Distribution of Edge Counts



Stable Distribution of Edge Counts



Thinned Model

Thinned Model

- Goal: match the $d_{0,1}$ and $d_{1,2}$ exponents separately (if $n \to \infty$)

Thinned Model

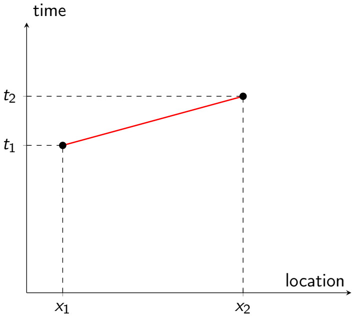
- Goal: match the $d_{0,1}$ and $d_{1,2}$ exponents separately (if $n \to \infty$)
- Observation: high edge degree $\Longrightarrow$ both endpoints are old

Thinned Model

- Goal: match the $d_{0,1}$ and $d_{1,2}$ exponents separately (if $n \to \infty$)
- Observation: high edge degree $\implies$ both endpoints are old
- Solution: remove edges that do not affect exponent of $d_{1,2}$

Thinned Model

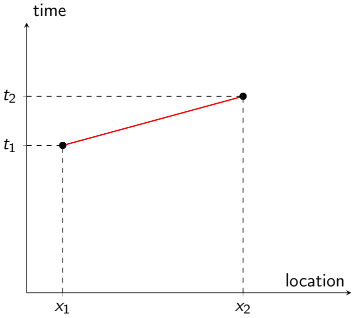
- Goal: match the $d_{0,1}$ and $d_{1,2}$ exponents separately (if $n \to \infty$)
- Observation: high edge degree $\implies$ both endpoints are old
- Solution: remove edges that do not affect exponent of $d_{1,2}$



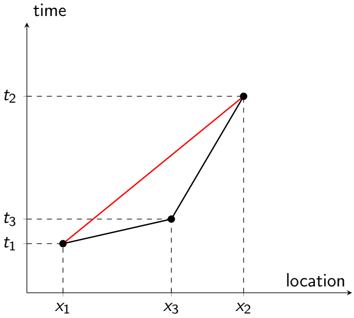
Protected: $t_1 \approx t_2$

Thinned Model

- Goal: match the $d_{0,1}$ and $d_{1,2}$ exponents separately (if $n \to \infty$)
- Observation: high edge degree $\Longrightarrow$ both endpoints are old
- Solution: remove edges that do not affect exponent of $d_{1,2}$



Protected: $t_1 \approx t_2$



Protected: $t_1 \approx t_3$

Thinned Model

- Remove exposed edges independently with probability $1 - t_1^{\eta}$

Thinned Model

- Remove exposed edges independently with probability $1 - t_1^{\eta}$
- $\lim_{k \to \infty} d_{0,1}(k) \sim k^{-\frac{1}{\eta - \gamma}}$

Thinned Model

- Remove exposed edges independently with probability $1 - t_1^{\eta}$
- $\lim_{k \to \infty} d_{0,1}(k) \sim k^{-\frac{1}{\eta - \gamma}}$
- $\lim_{k \to \infty} d_{1,2}(k) \sim k^{1 - \frac{2}{\gamma}}$

Q & A