On the Topology of Higher-Order Age-Dependent Random Connection Models

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Topology of Higher-Order Networks

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Higher-Order Networks





3 Age-Dependent Random Connection Model



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Publications of Authors in Statistics



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Results





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ADRCM

Results

Goal



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Age-Dependent Random Connection Model

Gracar, Peter, et al. "The age-dependent random connection model." *Queueing Systems* 93 (2019): 309-331.

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• Vertices: Poisson process $\mathcal{P}\{(x_i, t_i)\} \subset \mathbb{R} \times [0, 1]$

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Properties if $\gamma \in \left(\frac{1}{2}, 1\right)$:

- 🗸 sparse
- 🗸 scale-free
- 🗸 ultra small
- ✓ high clustering
- \times graphical model



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Research Outline

• Consider ADRCM as a "clique complex"

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Research Outline

- Consider ADRCM as a "clique complex"
- Determine characteristics of ADRCM as a higher-order network

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- Consider ADRCM as a "clique complex"
- Determine characteristics of ADRCM as a higher-order network
- Extend ADRCM to match a larger number of characteristics
- Illustrate the findings on simulated networks and on real data

ADRCM

Results

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Higher-Order Degree Distributions

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Higher-Order Degree Distributions



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Higher-Order Degree Distributions



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Higher-Order Degree Distribution – Statistics Theory



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Results

CLT for Betti Numbers



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Results

CLT for Betti Numbers





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CLT for Betti Numbers



• $\gamma \ll 1$ • $\frac{\beta_n - \mathbb{E}[\beta_n]}{\sqrt{\operatorname{Var}(\beta_n)}} \xrightarrow{d} \mathcal{N}(0, 1)$

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Distribution of Edge Count

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Distribution of Edge Count

CLT for Edge Count

- $\gamma < 1/2$
- S_n : number of *m*-simplices in the interval [0, *n*]

•
$$\frac{S_n - \mathbb{E}[S_n]}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma^2)$$

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Stable Limit Law for Edge Count

- $1/2 < \gamma < 1$
- *S_n* : number of edges in the interval [0, *n*]

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$$\frac{S_n - \mathbb{E}[S_n]}{n^{\gamma}} \xrightarrow{d} \mathcal{S}(1/\gamma)$$

Stable Distribution of Edge Counts



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Stable Distribution of Edge Counts



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Thinned Model

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• Goal: match the $d_{0,1}$ and $d_{1,2}$ exponents separately (if $n \to \infty$)

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- Goal: match the $d_{0,1}$ and $d_{1,2}$ exponents separately (if $n o \infty$)
- $\bullet\,$ Observation: high edge degree \Longrightarrow both endpoints are old

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Thinned Model

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- Solution: remove edges that do not affect exponent of $d_{1,2}$

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ADRCM

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Thinned Model

ullet Remove exposed edges independently with probability $1-t_1^\eta$

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Thinned Model

• Remove exposed edges independently with probability $1-t_1^\eta$

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$$\lim_{k\to\infty} d_{0,1}(k) \sim k^{-rac{1}{\eta-\gamma}}$$

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Thinned Model

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$$\lim_{k \to \infty} d_{0,1}(k) \sim k^{-rac{1}{\eta - \gamma}}$$

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$$\lim_{k\to\infty} d_{1,2}(k) \sim k^{1-\frac{2}{\gamma}}$$

Q & A

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