Topological Data Analysis of Higher-Order Networks

AARHUS UNIVERSITET

Christian Hirsch, Peter Juhasz

peter.juhasz@math.au.dk

Collaboration Networks

- Higher-order networks describe group interactions.
- Goal: build topological models to describe topological invariants of real-world higher-order networks.
- The models could shed light on the high-level structure of scientific collaborations.

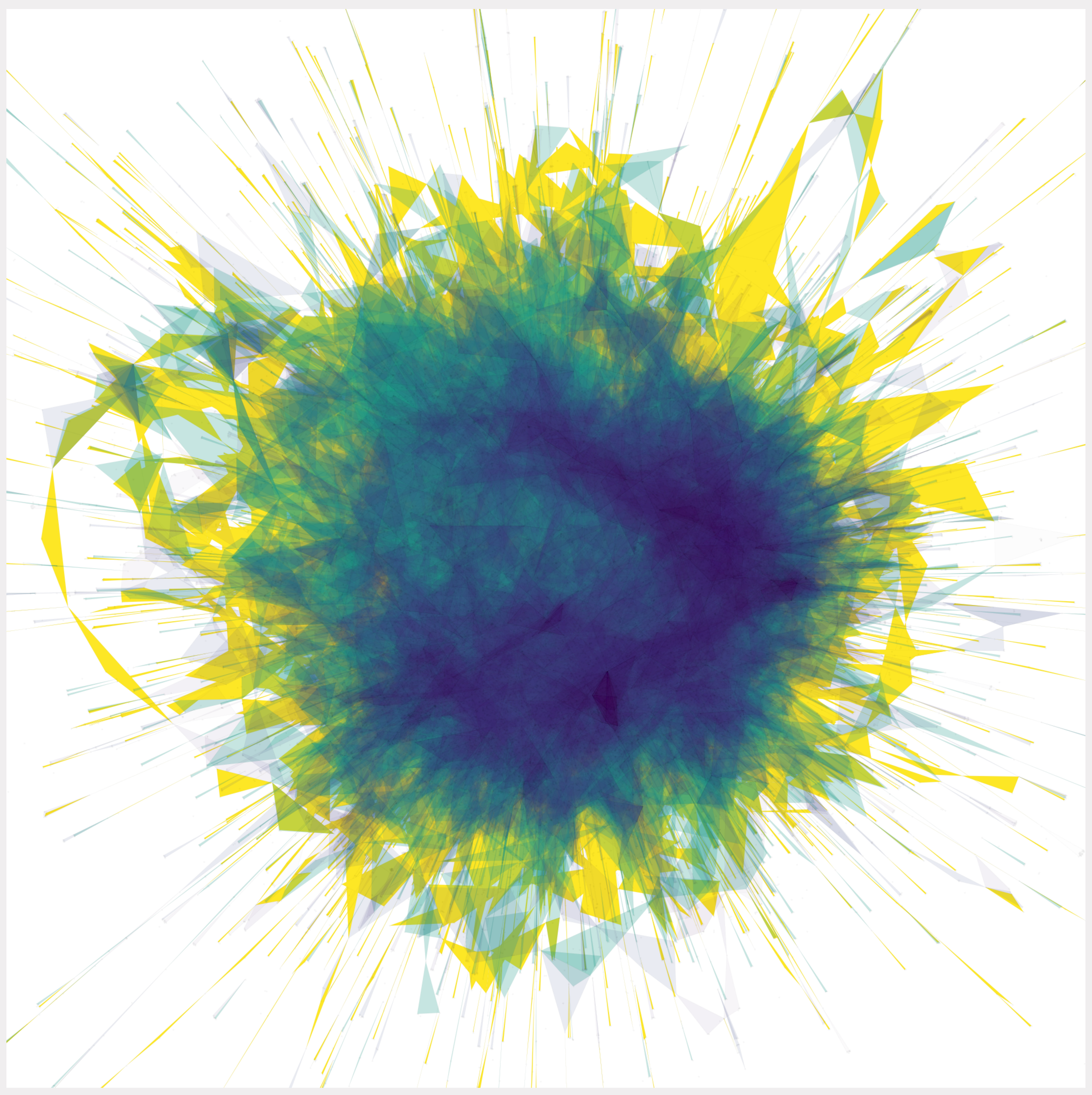

Figure 1: Collaboration of Scientists in the Field of Statistics

- Idea: use simplicial complexes where each interaction is represented by a simplex.
- Calculating Betti numbers describes high-level community structures.

Age-Dependent Random Connection Model

- The age-dependent random connection model [1] describes a growing graph.
- Vertices arrive according to a Poisson process $\mathcal{P} = \{(t_i, y_i)\}_{i \geq 1}$ at times t_i placed to a location y_i on a torus.
- A pair of nodes (t, y_i), $(\tau, y_j) \in \mathcal{P}$ is connected if

$$|y_i - y_j| \leq t^{-\gamma} \tau^{\gamma - 1} \qquad (t \leq \tau)\,; \; \gamma \in (0, 1),$$

- Result: preferential attachment + spatially induced clustering.

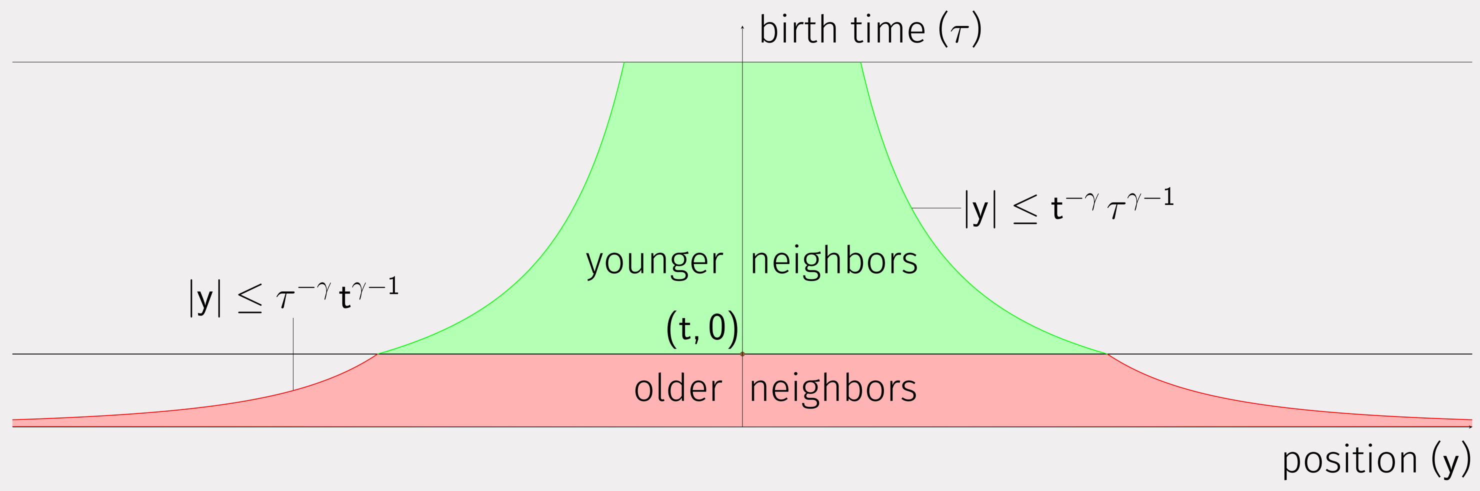

Figure 3: Simulation of a network. A vertex is placed to the origin with birth time t. The birth time τ of one of its neighbors is represented on the vertical axis. It connects to older nodes in the red shaded area, whereas younger nodes connect to it in the green shaded area of the graph.

- After a graph is created, it is expanded to a higher-order network by creating the clique complex of the graph.

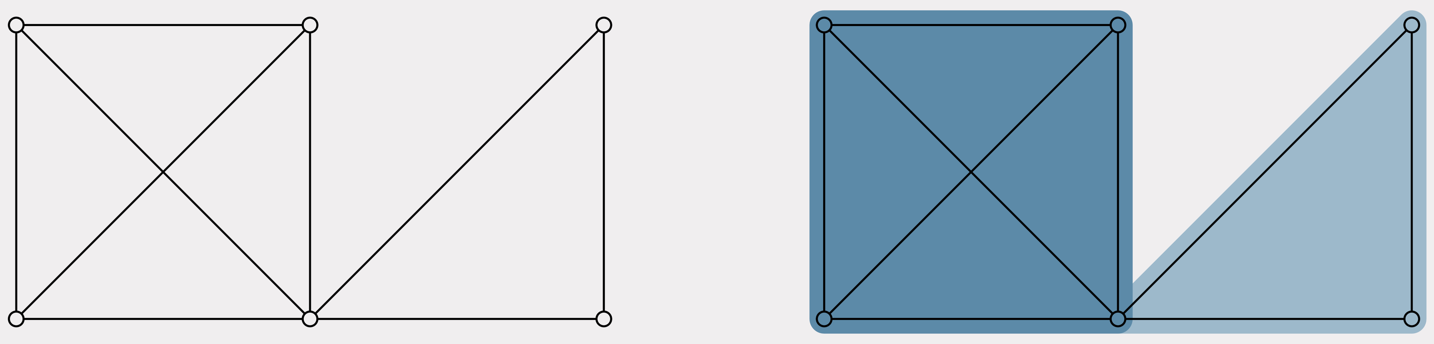

Figure 4: Expansion of a simple graph to a clique complex.

[1] P Gracar, A Grauer, L Lüchtrath, and P Mörters. The age-dependent random connection model. *Queueing Systems*, 93:309–331, 2019.

Normal Distribution of Betti Numbers

Theorem (Central limit theorem for Betti numbers). *Let* $q \geq 0$ *and* $\gamma < 1/4$. *Then,*

$$\frac{\beta_{n,q} - \mathbb{E}[\beta_{n,q}]}{\sqrt{\mathrm{Var}(\beta_{n,q})}} \xrightarrow{d} \mathcal{N}(0, 1).$$

- If $\gamma < 1/4$, the variance of the degree distribution is finite.
- The range of γ could be extended; conjecture: asymptotic normality breaks down if $\gamma \geq 0.5$.

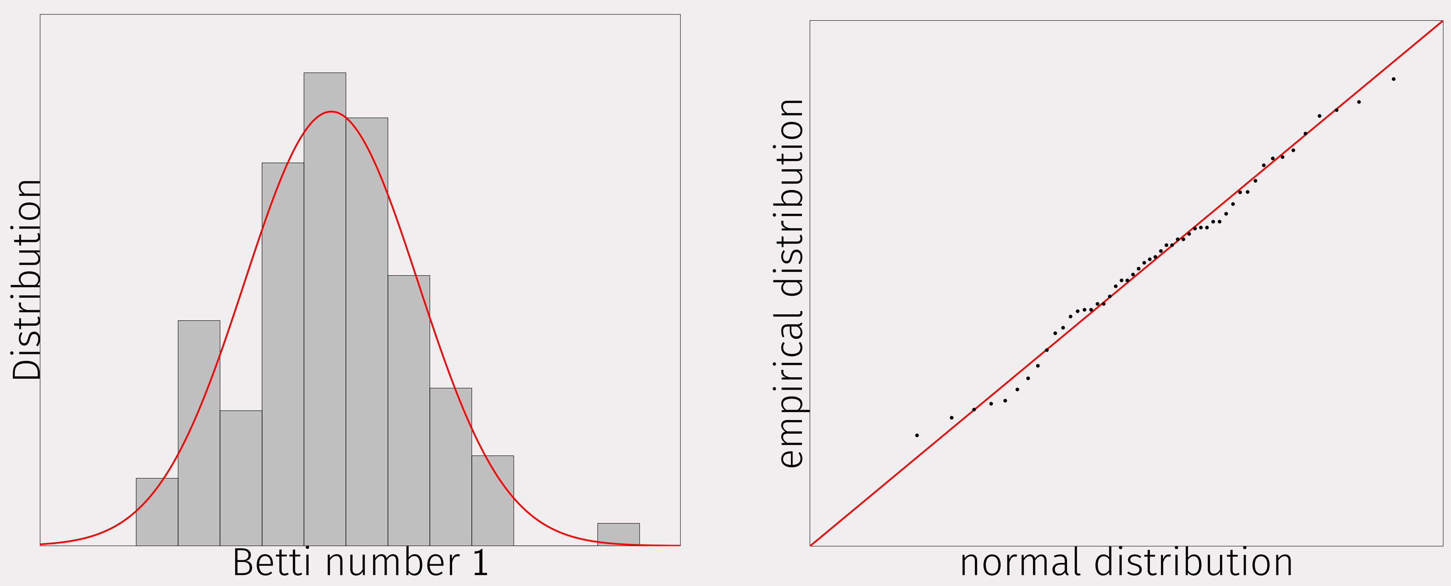

Figure 2: Left: normal distribution of Betti number 1 ($\gamma = 0.25$). Right: Q-Q plot with the fitted normal distribution.

Stable Distribution of Betti Numbers

Conjecture (Stable limit for Betti numbers). *Let* $q \geq 0$ *and* $\gamma > 1/2$. *Then,*

$$\frac{\beta_{n,q} - \mathbb{E}[\beta_{n,q}]}{n^{\alpha}} \xrightarrow{d} S\left(1/\gamma\right),$$

- For $\gamma = 3/4$, we simulated 100 networks of size $n = 10^5$.
- The Q-Q plot for Betti numbers shows a fat left tail of the distribution.

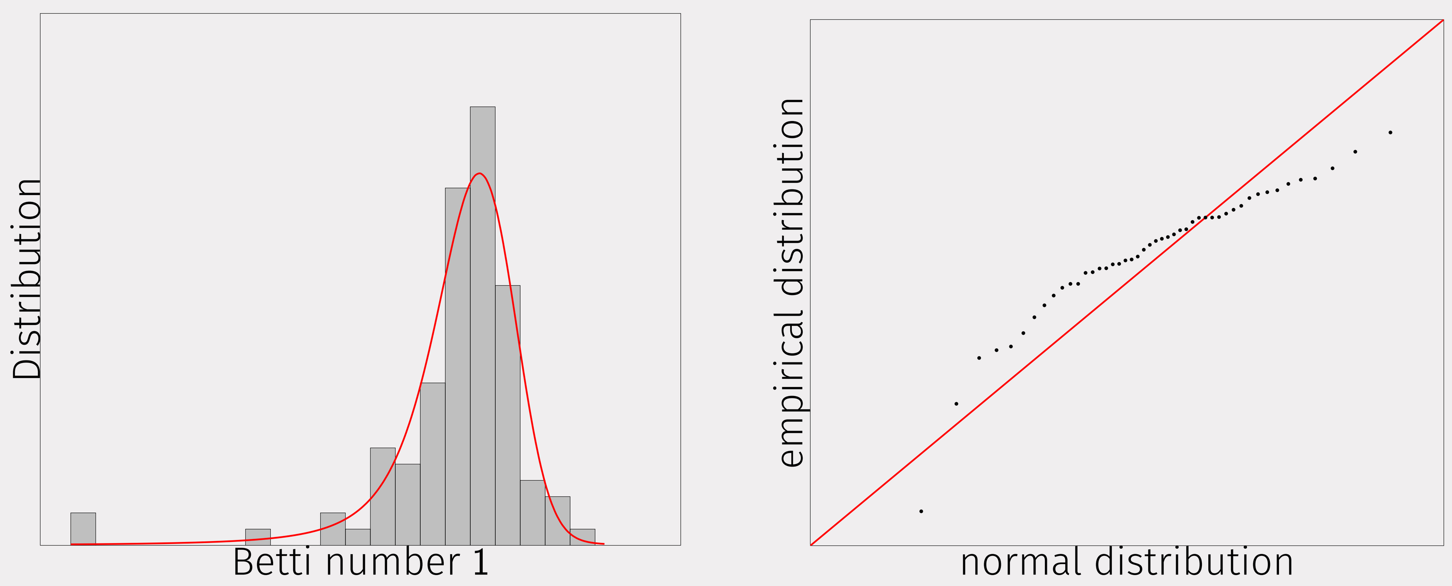

Figure 5: Left: stable distribution of Betti number 1 ($\gamma = 0.75$). Right: Q-Q plot with the fitted normal distribution.